

Explicit algebraic Reynolds stress model to predict anisotropy in shock-turbulence interaction

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Abstract

Shock waves drastically alter the nature of Reynolds stresses in a turbulent flow and conventional RANS models cannot reproduce this effect. In the present study, we evaluate Wallin and Johansson's explicit algebraic Reynolds stress model (EARSIM) against available DNS data for canonical shock turbulence interaction. The model is found to overpredict the post-shock Reynolds stresses for a range of Mach numbers. Budget of the transport equation of Reynolds stresses, computed using linear interaction analysis, shows that the unsteady shock distortion mechanisms and the pressure velocity correlations are important. We propose an improvement to the EARSIM to include these effects and redistribute the kinetic energy between the different normal Reynolds stresses. The new model is found to match DNS data for the amplification of Reynolds stresses across the shock and their post-shock evolution, for a range of Mach numbers.

1 Introduction

Shock-induced mixing plays a vital role in the hypersonic air-breathing engines, such as in combustion chamber of scramjets. The crucial part of the flow in the combustor is the turbulent mixing process of fuel and freestream supersonic air flow. Enhancement of turbulence levels due to shock increases the mixing efficiency. Shock amplifies the Reynolds stresses in the streamwise and the transverse directions differently. Capturing these complex flow-features is challenging for conventional turbulence models and the evaluation of turbulent kinetic energy and Reynolds stress anisotropy needs reliable turbulence models.

The canonical shock/turbulence interaction (Fig. 1) can illustrate the fundamental flow phenomenon and provides physical insight of the turbulent mixing effectiveness. Homogeneous isotropic turbulence passing through a nominally normal shock is possibly the most fundamental shock-turbulence interaction (STI), shown in Fig. 1. The mean flow is one-dimensional, steady and is uniform upstream and downstream of the shock wave. The shock wave amplifies the turbulent fluctuations, and the amplification can be highly anisotropic. The jump in the mean flow quantities across the shock is governed by the Rankine-Hugoniot relations. The detail description of the canonical shock-turbulence interaction is given in [1].

Larsson *et al.* [3] presents DNS of the STI, which shows a good match for turbulence kinetic energy (TKE) against linear interaction analysis (LIA) data. The Reynolds stresses and its anisotropy behind the shock, however, do not match between DNS and LIA. For low values of turbulent Mach number (M_t), the DNS data of Ryu and Livescu [4] show that the amplification of the Reynolds stresses approach the LIA results. Schwarzkopf *et al.* [5] proposed a second-moment closure model for variable density flows and applied the model to homogeneous shock-turbulence interaction. Braun *et al.* [6] extended the model

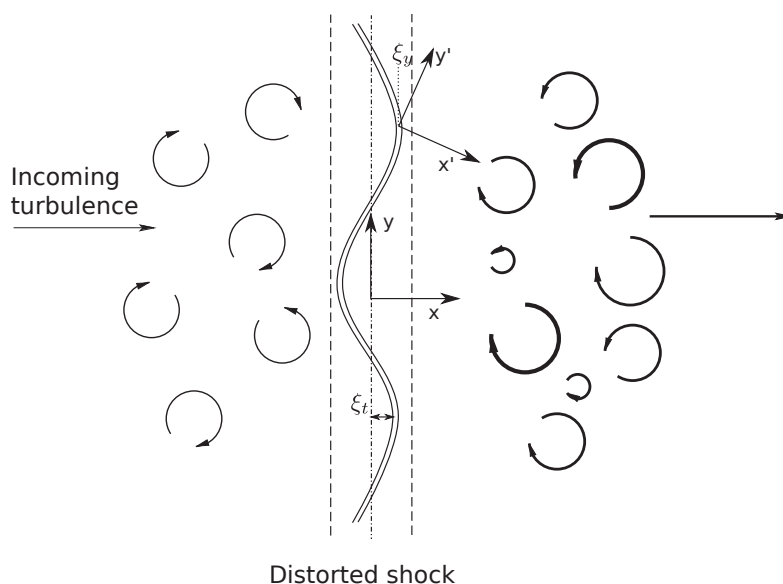


Fig. 1 Schematic of homogeneous isotropic turbulence/shock interaction.

of Schwarzkopf *et al.* [5] to investigate the Reynolds stresses and its anisotropy for the shock/turbulence interaction problem. Braun *et al.* [6] model matches reasonably with LIA predictions. The individual Reynolds stress amplifications predicted by these models, however have shown limited agreement with DNS data. Griffond *et al.* [7] modified the Gregoire *et al.* [8] Reynolds stress model to match results for shock/turbulence interactions. Following the same approach Griffond and Soulard [9] evaluated three augmented Reynolds stress models which have additional transport equation for the density related correlation against LIA data. Similarly, Vemula and Sinha [1] also modified the Reynolds stress model of Gerolymos *et al.* [10] with LIA predictions and presented a detailed study of the turbulent Reynolds stresses across the shock wave.

Models based on Reynolds stress transport equations are found to be better compared to conventional models such as Spalart-Allmaras (SA), $k-\epsilon$ and $k-\omega$. When an isotropic turbulence, which is solenoidal in nature passes through a normal shock wave, the turbulence gets compressed and becomes anisotropic in nature. In standard two-equation models, the eddy-viscosity assumption fails to predict the Reynolds stresses (such as, $\overline{u_i''u_j''} = R_{ij} = 2/3k\delta_{ij}$) and yields very high amplification of the turbulence across the shock [13]. On the other hand, the explicit algebraic Reynolds stress model (EARSIM), computes the Reynolds stresses using the non-linear relation between the Reynolds stresses and mean strain. In EARSIM, the anisotropy tensor ($a_{ij} = R_{ij}/k - 2\delta_{ij}/3$) is written as a non-linear combination of the strain rate tensor, S_{ij} and the rotation tensor, Ω_{ij} , which allows a direct evaluation of the Reynolds stress terms once the mean flow is known.

The primary purpose of this study is to capture the Reynolds stresses and its anisotropy generated across the shock using the EARSIM turbulence model. The base EARSIM model that is considered in this work is that of Wallin and Johansson's EARSIM [2], whose performance is evaluated in the next section for the case of an isotropic homogeneous turbulence interacting with a nominally normal shock. The limitations of the base model are presented, and appropriate corrections are described. Section 3 derives a new set of algebraic equations for the Reynolds stresses behind the shock using the linear analysis. The improved model is generalized and compared against DNS data in section 4.

2 Explicit algebraic Reynolds stress model

In the present work, the Wallin and Johansson's EARSM formulation is used for calculating the Reynolds stress tensor, which is expressed in terms of the effective turbulent viscosity and an extra anisotropy. It is given as,

$$R_{ij} = \frac{2}{3}\rho k\delta_{ij} - 2\mu_T S_{ij}^* + \rho k a_{ij}^{ext}. \quad (1)$$

Here, μ_T and a_{ij}^{ext} are written as, $-\frac{1}{2}(\beta_1 + II_\Omega\beta_6)\rho k\tau$ and $\rho k [\beta_4(S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj})]$ respectively. The dimensional strain- and vorticity rate tensors are given as,

$$S_{ij}^* = \frac{1}{2} \left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} - \frac{2}{3} \frac{\partial \tilde{u}_k}{\partial x_k} \delta_{ij} \right), \text{ and } \Omega_{ij}^* = \frac{1}{2} \left(\frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i} \right). \quad (2)$$

The above quantities are non-dimensionalized as,

$$S_{ij} = \tau S_{ij}^*, \text{ and } \Omega_{ij} = \tau \Omega_{ij}^*. \quad (3)$$

Here, τ is the time scale with a Kolmogorov limiter (ratio of the TKE and dissipation rate). Coefficients of the above tensors β_i are defined as,

$$\beta_1 = -\frac{N(2N^2 - 7II_\Omega)}{Q}, \beta_4 = -\frac{2(N^2 - 2II_\Omega)}{Q}, \text{ and } \beta_6 = -\frac{6N}{Q},$$

where Q is calculated as,

$$Q = \frac{5}{6}(N^2 - 2II_\Omega)(2N^2 - II_\Omega)$$

and N is the solution of a cubic polynomial equation for two-dimensional mean flow and is given as,

$$N = \begin{cases} \frac{C'_1}{3} + (P_1 + \sqrt{P_2})^{1/3} + \text{sgn}(P_1 - \sqrt{P_2})|P_1 - \sqrt{P_2}|^{1/3}, & \text{at } P_2 \geq 0 \\ \frac{C'_1}{3} + 2(P_1^2 - P_2)^{1/6} \cos\left(\frac{1}{3} \arccos\left(\frac{P_1}{\sqrt{P_1^2 - P_2}}\right)\right), & \text{at } P_2 < 0 \end{cases}$$

with

$$P_1 = C'_1 \left(\frac{C'_1}{27} + \frac{9}{20}II_S - \frac{2}{3}II_\Omega \right), \text{ and } P_2 = P_1^2 - \left(\frac{C'^2_1}{9} + \frac{9}{10}II_S - \frac{2}{3}II_\Omega \right)^3. \quad (4)$$

The tensor invariants II_S and II_Ω are written as a function of strain and vorticity tensors, and are given as,

$$II_S = S_{ij}S_{ji}, \text{ and } II_\Omega = \Omega_{ij}\Omega_{ji}. \quad (5)$$

Standard $k - \epsilon$ [11] or $k - \omega$ [12] turbulence model can be used as the base model for computing the TKE and its dissipation rate. The complete formulation and the detailed procedure to obtain the β coefficients are provided by Wallin and Johansson [2].

For the case of STI, the mean velocity gradients are finite in the streamwise direction only, hence the value of Ω_{ij} is zero. Therefore, the coefficient β_6 and the extra anisotropy, a_{ij}^{ext} terms in the Reynolds stress tensor becomes zero and the streamwise- and transverse Reynolds stresses are simplified to

$$R_{11} = \frac{2}{3}\rho k\delta_{11} + \rho k\beta_1\tau S_{11}^*, \quad (6)$$

$$R_{22} = \frac{2}{3}\rho k\delta_{22} + \rho k\beta_1\tau S_{22}^*, \quad (7)$$

and the effective turbulent viscosity, μ_T becomes $0.5\beta_1\rho k\tau$ with $S_{11}^* = 2\partial u_1/\partial x_1$, $S_{22}^* = -\partial u_1/\partial x_1$, $\tau = k/\epsilon$, and $\beta_1 = -6/(5N)$. For the STI case P_2 value is less than zero and the solution for the term N is obtained using following expression

$$N = \frac{c'_1}{3} + 2(P_1^2 - P_2)^{1/6} \cos\left(\frac{1}{3} \arccos\left(\frac{P_1}{\sqrt{P_1^2 - P_2}}\right)\right), \quad (8)$$

with $P_1 = c'_1\left(\frac{c_1'^2}{27} + \frac{9}{20}II_S\right)$, $P_2 = P_1^2 - \left(\frac{c_1'^2}{9} + \frac{9}{10}II_S\right)^3$, $c'_1 = 1.8$ and $II_S = 2\tau^2(\partial u_1/\partial x_1)^2/3$. Here, N , β_1 , R_{11} , and R_{22} are functions of the mean strain rate, TKE and its dissipation rate. By considering the II_S contribution only, the solution for N reduces to $1.8(P_1^2 - P_2)^{1/6}$, and the approximate values of the N and the coefficient β_1 are $1.39\tau|\partial\tilde{u}/\partial x|$ and $-0.86/(\tau\partial\tilde{u}/\partial x)$, respectively (see Appendix). Substituting the β_1 value in the Reynolds stress tensor equation gives the resultant normal Reynolds stresses, which are depend on TKE only.

$$\begin{aligned} R_{11} &= \frac{2}{3}\rho k + 0.573\rho k, \\ R_{22} &= \frac{2}{3}\rho k - 0.286\rho k. \end{aligned}$$

In the present work, standard $k - \epsilon$ [11] turbulence model is used as the base model for calculating TKE and its dissipation rate. For a one-dimensional mean flow, the transport equation for TKE is simplified as,

$$\bar{\rho}\tilde{u}\frac{\partial k}{\partial x} = -R_{11}\frac{\partial\tilde{u}}{\partial x} - \bar{\rho}\epsilon. \quad (9)$$

By neglecting the dissipation rate term at the shock, a closed-form solution for the amplification of TKE across the shock is obtained as,

$$\frac{k_d}{k_u} = \left(\frac{\tilde{u}_u}{\tilde{u}_d}\right)^{2/3+0.573}. \quad (10)$$

Similarly, the normal Reynolds stresses are obtained as,

$$\frac{R_{11,d}}{R_{11,u}} = 1.86 \left(\frac{\tilde{u}_u}{\tilde{u}_d}\right)^{2/3+0.573}, \quad (11)$$

$$\frac{R_{22,d}}{R_{22,u}} = 0.57 \left(\frac{\tilde{u}_u}{\tilde{u}_d}\right)^{2/3+0.573}. \quad (12)$$

with subscripts u and d denoting the locations upstream and downstream of the shock, respectively.

Fig. 2 presents the comparison of the closed-form solution of EARSIM to predict the amplification of TKE, the Reynolds stresses and its anisotropy across the normal shock wave with the DNS data for varying shock strengths. EARSIM yields large amplification of TKE, over-predicting the DNS data by a factor of 2 at 1.87 and more at large Mach number. Due to rapid compression of the turbulence, the effective turbulent viscosity across the shock wave is highly over-predicted, which leads to high amplification of TKE. Additionally, EARSIM does not capture the anisotropy variation with Mach number and gives a constant anisotropy of value 3.26 for varying shock strengths.

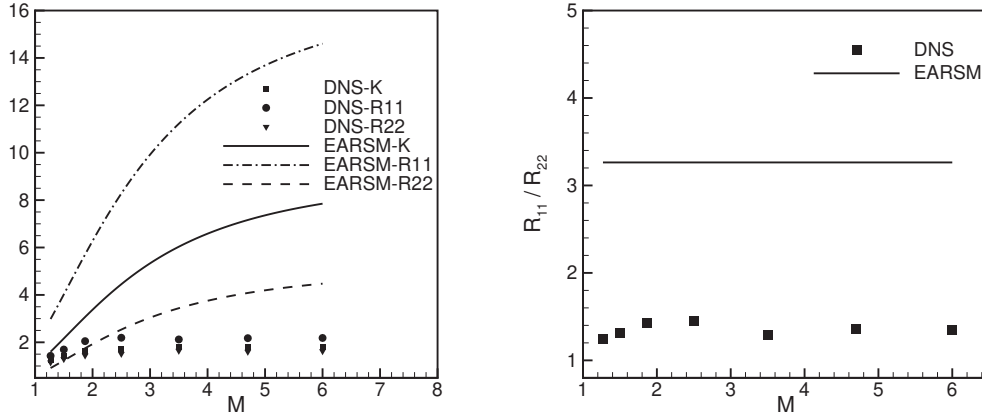


Fig. 2 Comparison of the amplification in the TKE, Reynolds stresses and its anisotropy across the normal shock obtained using Wallin and Johansson's EARS-M with DNS data for varying upstream Mach number.

3 LIA based model improvement

LIA is a theoretical tool, where the shock wave is treated as an inviscid discontinuity and the turbulence field upstream and downstream of the shock are decomposed as Fourier modes. The fluctuating velocity field in a turbulent flow interacts with the normal shock, making the shock oscillatory and unsteady in nature. To quantify the effect of the shock unsteadiness on the turbulent quantities, Sinha *et al.* [13] derived a transport equation for TKE in the frame of reference attached to the instantaneous shock wave.

$$\overline{\rho u} \frac{\partial k}{\partial x} = P_k + S_k^1 + 2S_k^2 + \Pi_k^1 + \Pi_k^2 + \Pi_k^3. \quad (13)$$

The terms on the right side of Eq. (13) are in order, production due to mean compression, shock-unsteadiness, shock-distortion, velocity pressure correlation, change in TKE due to acoustic energy transfer and production due to the mean pressure gradient.

The transport equations for the individual Reynolds stresses based on the reference frame attached to the instantaneous shock can be written as,

$$\overline{\rho u} \frac{\partial}{\partial x} \frac{R_{11}}{2} = P_k + S_k^1 + \Pi_k^1 + \Pi_k^2 + \Pi_k^3, \quad (14)$$

$$\overline{\rho u} \frac{\partial}{\partial x} \frac{R_{22}}{2} = S_k^2, \quad (15)$$

$$\overline{\rho u} \frac{\partial}{\partial x} \frac{R_{33}}{2} = S_k^2. \quad (16)$$

By combining the above equations we get TKE equation, Eq. (13). For the purely vortical turbulence, the upstream fluctuation, such as $\rho'_u = p'_u = T'_u$ are zero, hence Π_k^3 is zero. Just immediately after the shock wave the vortical fluctuations are redistributed into acoustic fluctuations with the finite values for $\overline{p'^2}$ and $\overline{p'u'}$ correlations. This adjustment happens within one characteristic eddy size, where the acoustic energy decays and transferred back to the TKE. The change in TKE due to this energy transfer mechanism is denoted by a source term, Π_k^2 . For further details of the LIA and the budget analysis, see Refs. [13] & [14].

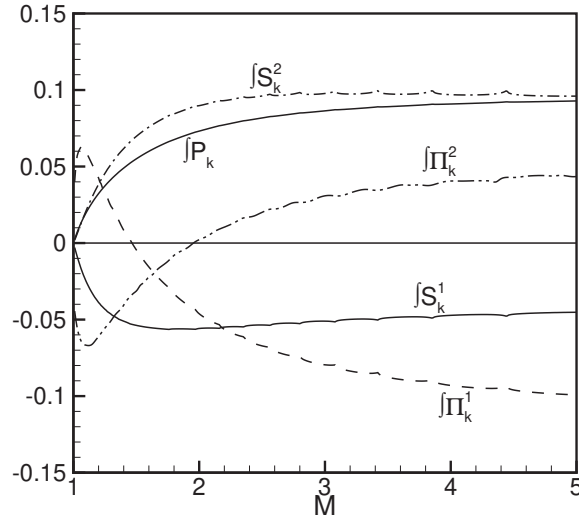


Fig. 3 Budget of Reynolds stress equations at different Mach numbers. All terms are normalized by $\overline{\rho_1} \overline{u_1^3}$

From the Rankine-Hugoniot relations [13], we can write the linearized equations accounting in the change of R_{11} , R_{22} , and R_{33} across the shock as,

$$\frac{1}{2} \overline{\rho_u} \overline{u_u} (R_{11,d} - R_{11,u}) = -\overline{\rho_u} \overline{u'_u u'_m} \Delta \overline{u} + \overline{\rho_u} \overline{u'_m \xi_t} \Delta \overline{u} - \overline{u'_m (p'_d - p'_u)}, \quad (17)$$

$$\frac{1}{2} \overline{\rho_u} \overline{u_u} (R_{22,d} - R_{22,u}) = -\overline{\rho_u} \overline{u_u v'_m \xi_y} \Delta \overline{u}, \quad (18)$$

$$\frac{1}{2} \overline{\rho_u} \overline{u_u} (R_{33,d} - R_{33,u}) = -\overline{\rho_u} \overline{u_u w'_m \xi_z} \Delta \overline{u}, \quad (19)$$

where $\Delta \overline{u} = (\overline{u_d} - \overline{u_u})$, $u'_m = (u'_u + u'_d)/2$, $v'_m = (v'_u + v'_d)/2$, and $w'_m = (w'_u + w'_d)/2$. We assumed that $w' = v'$ and for small shock distortion, $\overline{w' \xi_z} = \overline{v' \xi_y}$. Here, overbar represents the Reynolds averaged, prime represents the fluctuation quantities, p is the pressure, ξ_t is the linear shock velocity in the streamwise direction, ξ_y and ξ_z are angular distortion of the shock.

The budget of the terms from the linearized transport equations obtained by using LIA shows that production due to mean compression amplifies the R_{11} , while the shock unsteadiness term reduces the magnitude of shock-normal Reynolds stresses. The shock distortion mechanism found to amplify R_{22} and R_{33} across the shock, which is opposite to the effect of the shock-unsteadiness term. Similar trend is seen in the pressure dependent terms i.e., Π_k^1 and Π_k^2 . Overall, the distortion of the shock front and its unsteady movement of the shock wave, and the acoustic energy transfer are responsible for redistribution of the TKE in three directions.

Amplification of the Reynolds stresses across the shock can be written in a simplified

form by rearranging Eqs. (17)-(19).

$$\frac{R_{11,d}}{R_{11,u}} = 1 - C_p \frac{\Delta \bar{u}}{\bar{u}_u} + C_u \frac{\Delta \bar{u}}{\bar{u}_u} + C_{\Pi} \frac{\Delta \bar{u}}{\bar{u}_u}, \quad (20)$$

$$\frac{R_{22,d}}{R_{22,u}} = 1 - C_d \frac{\Delta \bar{u}}{\bar{u}_u}, \quad (21)$$

$$\frac{R_{33,d}}{R_{33,u}} = 1 - C_d \frac{\Delta \bar{u}}{\bar{u}_u}, \quad (22)$$

$$\frac{k_d}{k_u} = 3 - C_p \frac{\Delta \bar{u}}{\bar{u}_u} + C_u \frac{\Delta \bar{u}}{\bar{u}_u} + C_{\Pi} \frac{\Delta \bar{u}}{\bar{u}_u} - 2C_d \frac{\Delta \bar{u}}{\bar{u}_u}. \quad (23)$$

Here, the coefficients are written as, $C_p = 2\overline{u'_u u'_m} / \overline{u'^2_u}$, $C_u = 2\overline{u'_m \xi_t} / \overline{u'^2_u}$, $C_{\Pi} = 2\overline{u'_m p'_d} / \overline{u'^2_u} \Delta \bar{u} + \Pi^2_k \bar{u}_u / \Delta \bar{u}$, and $C_d = 2\overline{v'_m \xi_y} / \overline{v'^2_u}$. Larrson *et al.* [3] shows that the amplification of the Reynolds stresses obtained from the LIA do not match with the DNS data, whereas for TKE, the LIA and the DNS data match very well (see Fig. 5).

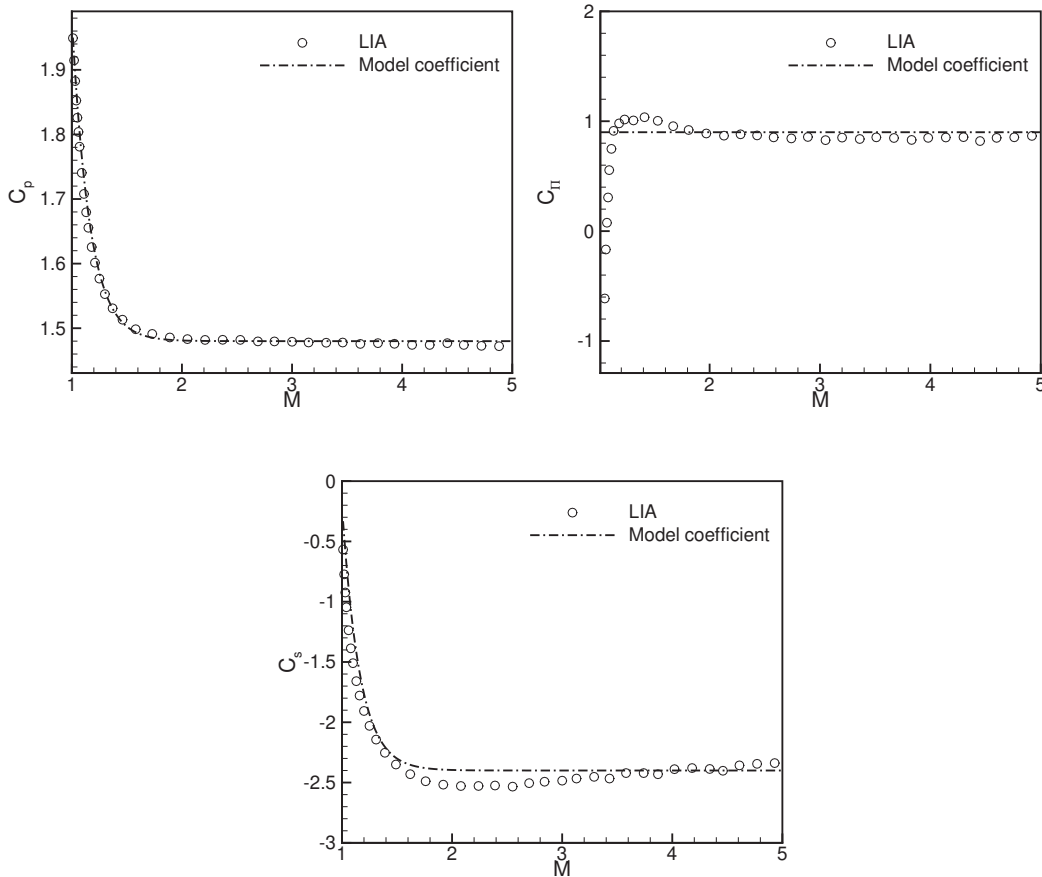


Fig. 4 The ratios predicted by linear analysis are used to obtain the model coefficients C_p , C_{II} and C_s .

In order to improve the prediction of the amplification of the Reynolds stresses, new algebraic Reynolds stress equations are proposed based on the observations from linear

analysis. The modified algebraic equations are,

$$\frac{R_{11,d}}{R_{11,u}} = 1 - C_p \frac{\Delta \bar{u}}{\bar{u}_u} + \left[\frac{1}{3} C_s + \frac{2}{3} C_\Pi \right] \frac{\Delta \bar{u}}{\bar{u}_u}, \quad (24)$$

$$\frac{R_{22,d}}{R_{22,u}} = 1 + \left[\frac{1}{3} C_s + \frac{1}{6} C_\Pi \right] \frac{\Delta \bar{u}}{\bar{u}_u}, \quad (25)$$

$$\frac{R_{33,d}}{R_{33,u}} = 1 + \left[\frac{1}{3} C_s + \frac{1}{6} C_\Pi \right] \frac{\Delta \bar{u}}{\bar{u}_u}, \quad (26)$$

$$\frac{k_d}{k_u} = 3 - C_p \frac{\Delta \bar{u}}{\bar{u}_u} + C_u \frac{\Delta \bar{u}}{\bar{u}_u} + C_\Pi \frac{\Delta \bar{u}}{\bar{u}_u} - 2C_d \frac{\Delta \bar{u}}{\bar{u}_u}. \quad (27)$$

The coefficients C_p , C_u , C_Π and C_d predicted by linear analysis for varying Mach number are shown in Fig. 4 and the curve-fits are

$$\begin{aligned} C_p &= 1.48 + 0.5e^{(6.2(1-M_{1n}))}, \\ C_\Pi &= 0.9, \\ C_s &= C_u - 2C_d = -2.5 + 2.2e^{(6.2(1-M_{1n}))}. \end{aligned}$$

Fig. 5 shows the comparison of the amplification of TKE and the Reynolds stresses, and its anisotropy across the shock obtained using the modified algebraic equations against LIA and the DNS data. The TKE obtained from LIA shows a better match with the DNS data, whereas the streamwise and transverse Reynolds stress are under- and over-predicted, respectively in comparison to the DNS data. This is due to the effect of the shock-unsteadiness term, which would reduce the streamwise Reynolds stress and the shock-distortion term will result in a much larger amplification for the transverse Reynolds stress. In modified algebraic Eqs. (24)-(27), the overall effect of shock oscillation and acoustic energy exchange are redistributed among the Reynolds stresses by retaining TKE across the shock. Therefore, the Reynolds stresses predicted by the modified algebraic equations match better with the DNS data as shown in Fig. 5.

4 Model evaluation

Using LIA, Sinha *et al.* [13] developed the shock unsteadiness (SU) $k - \epsilon$ model and the model found to predict the jump in TKE accurately across the shock. Applying to realistic flows involving multi-dimensional, complex shock structure, and shock-boundary layer interaction, the SU $k - \epsilon$ model has shown good potential. In the present study, SU $k - \epsilon$ model is used as the base model and the transport equations in a general tensor form are written as,

$$\frac{\partial \rho k}{\partial t} + \frac{\partial \rho \tilde{u}_j k}{\partial x_j} = -\frac{2}{3}(1 - b'_1) \rho k \frac{\partial \tilde{u}_i}{\partial x_i} - \rho \epsilon, \quad (28)$$

$$\frac{\partial \rho \epsilon}{\partial t} + \frac{\partial \rho \tilde{u}_j \epsilon}{\partial x_j} = -\frac{2}{3} \rho \epsilon C_{\epsilon 1} \frac{\partial \tilde{u}_i}{\partial x_i} - C_{\epsilon 2} \rho \frac{\epsilon^2}{k}. \quad (29)$$

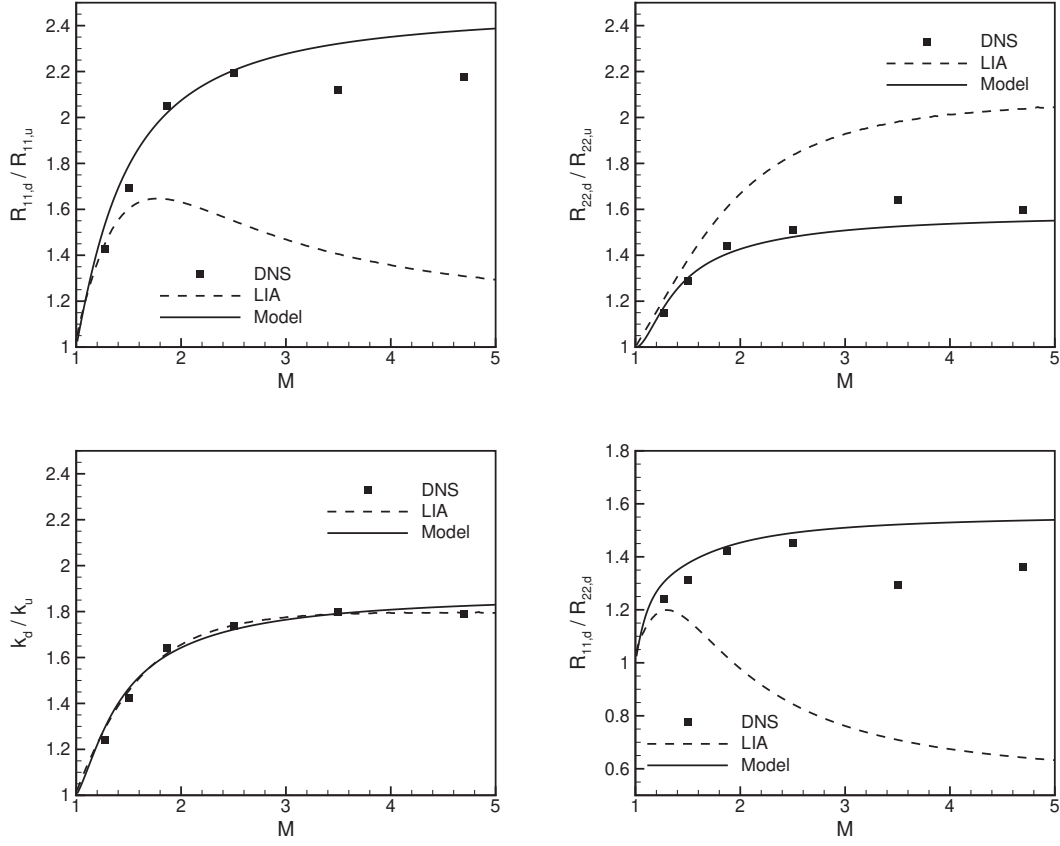


Fig. 5 Comparison of the amplification of TKE, the Reynolds stresses and its anisotropy across the normal shock as a function of upstream Mach number against the DNS data and LIA farfield data.

Here, $b'_1 = 0.4 (1 - e^{(1-M_{1n})})$, $C_{\epsilon 1} = 1.42$, and $C_{\epsilon 2} = 1.2$ are the model constants. The Reynolds stress anisotropy, a_{ll} is given by Eqs. (24)-(25),

$$a_{ll} = \frac{\left[1 - C_p \frac{\Delta \bar{u}}{\bar{u}_u} + \left(\frac{1}{3}C_s + \frac{2}{3}C_{\Pi}\right) \frac{\Delta \bar{u}}{\bar{u}_u}\right]}{\left[1 + \left(\frac{1}{3}C_s + \frac{1}{6}C_{\Pi}\right) \frac{\Delta \bar{u}}{\bar{u}_u}\right]}, \quad (30)$$

$$(31)$$

and the Reynolds stresses are computed as,

$$R_{22} = \frac{2k}{a_{ll} + 2}, \quad (32)$$

$$R_{11} = a_{ll} R_{22}. \quad (33)$$

Eqs. (28)-(29) have a form similar to the mean flow equations, and they can be solved either in a coupled or decoupled manner in a RANS code. The model equations are solved in the regions of high compression, i.e., shock waves, and everywhere else in the flowfield, a conventional EARSM turbulence model is solved. In this work, we follow the methodology adopted in earlier works involving the shock-unsteadiness $k - \epsilon$ model [13], [15].

Larsson *et al.* [3] present DNS data of the canonical STI problem for Mach numbers from 1.27 to 6 with varying turbulent Mach number, M_t , and Reynolds number based on Taylor microscale, Re_λ . The case with M_t in the range of 0.15–0.22, and $Re_\lambda = 40$ are considered for the present study. Normalized values of the Reynolds stresses, TKE, and its dissipation rate at the inlet station are calculated from the DNS data using

$$k_u = \frac{M_t^2}{2}, \quad \epsilon_u = \frac{5M_t^3}{\sqrt{3}\kappa_0\lambda^*Re_\lambda}, \quad \text{and} \quad R_{11,u} = R_{22,u} = \frac{2}{3}k_u, \quad (34)$$

with $\kappa_0\lambda^* = 0.84$. The Taylor micro-scale is given by $\lambda^* = \sqrt{10\nu^*k^*/\epsilon^*}$ and the Reynolds number based on λ^* is defined as $Re_\lambda = \lambda^*u_{rms}^*/\nu^*$. The turbulence kinetic energy and its dissipation rate are normalized as $k = k^*/a_\infty^{*2}$, and $\epsilon = \epsilon^*/a_\infty^{*3}\kappa_0$. Here, a_∞^* is the freestream speed of sound. The shock-upstream values are extrapolated to the inlet station using the homogeneous isotropic turbulence decay relations [16]. The diffusive fluxes are not included in the computation because their contribution is expected to be small [17].

Figs. 6 & 7 show the comparison of the streamwise evolution of the turbulent quantities obtained using the new model and the DNS data for the Mach numbers within the range of 1.5 to 6.0. For the homogeneous isotropic turbulence passing through normal shock, the change in mean velocity across the shock determines the jump in the turbulent quantities and the dissipation rate, ϵ determine the pre- and post-shock decay of the Reynolds stresses and TKE. The anisotropy in the dissipation rate tensor has a significant influence on the evolution of the Reynolds stresses and the TKE. In the second-moment closure models, the slow pressure-strain term brings the effect of the anisotropy in the dissipation rate [1], whereas in the present work, this effect is implemented in the new model using a_{II} .

The amplification in R_{11} , R_{22} , and TKE across the shock wave are reproduced well by the new model. The DNS data shows large values of R_{11} and R_{22} in the shock region and a non-monotonic variation of R_{11} immediately after the shock. This non-monotonic behavior is due to the acoustic decay, and this phenomenon is explained in Ref. [1] in detail. To estimate the effective jump in the turbulent quantities across the shock, the post-shock DNS data is extrapolated back to the shock-center location ($x = 0$) and is shown by an open circle. The post-shock evolution of the streamwise Reynolds stresses obtained using the new model decays monotonically and has similar decay rate as that of the DNS for all the Mach numbers. Both the DNS and the new model give similar predictions in the amplification and anisotropy in the Reynolds stresses across the shock, where $R_{11} > R_{22}$ for all Mach numbers.

5 Conclusion

We use numerical and theoretical tools to study the variation in the Reynolds stresses and the shock induced anisotropy for the case of homogeneous isotropic turbulence interacting with a nominally normal shock wave. To capture the anisotropy across the shock, the Wallin and Johansson's EARSMD turbulence model are simplified for a steady one-dimensional mean flow, which are in turn solved to get the jump in the normal Reynolds stresses across the shock. It is found that EARSMD predicts very high values of the Reynolds stresses against the DNS data.

The budget of the terms from the linearized transport equations obtained from LIA are used to improve the effective turbulent viscosity, thereby resulting in better predictions of Reynolds stresses and its anisotropy across the shock wave. The new LIA based algebraic

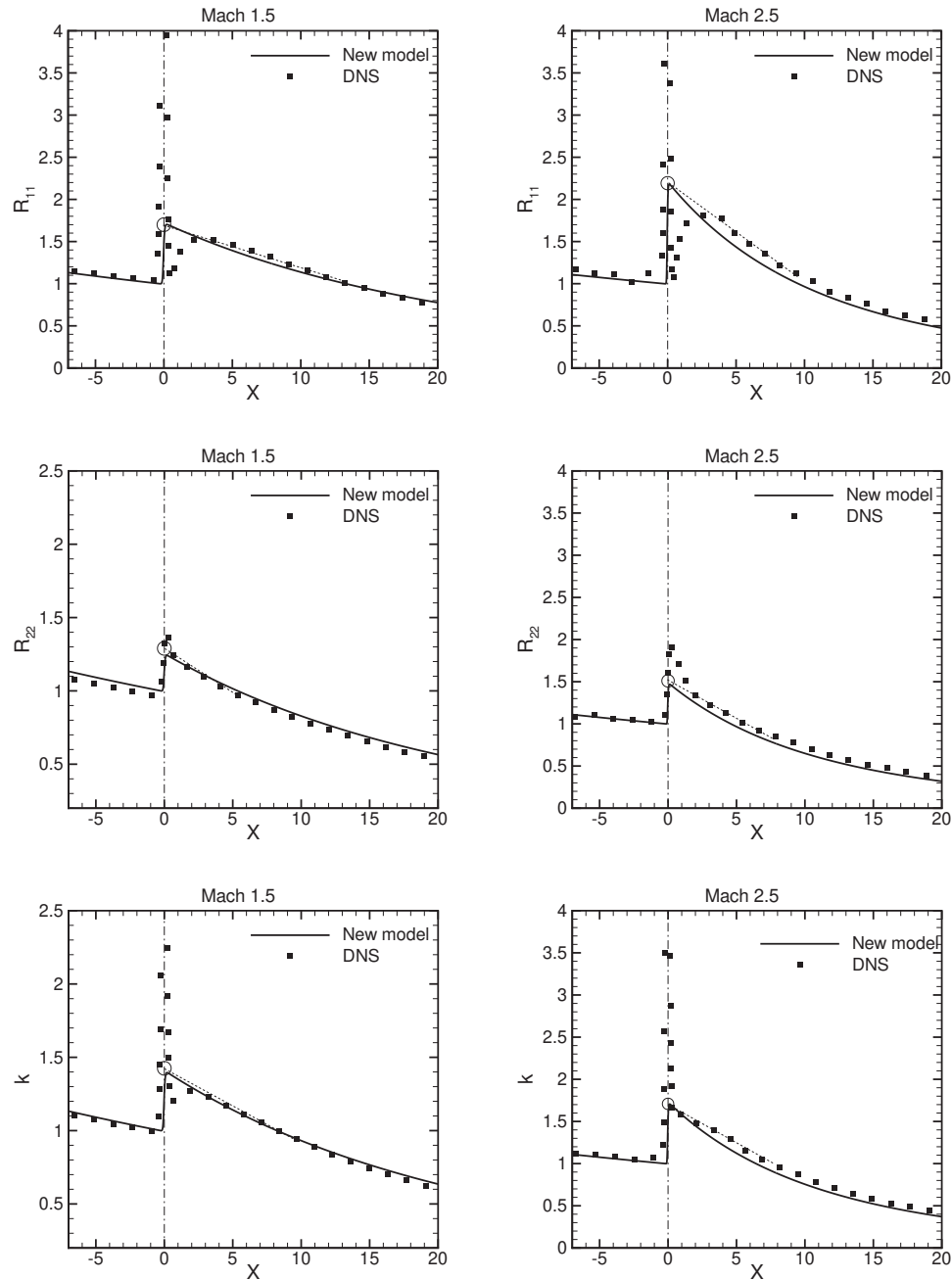


Fig. 6 Comparison of spatial variation of the normalized R_{11} , R_{22} , and TKE obtained using the new model against the DNS data at $M = 1.5$ and 2.5 .

equation for Reynolds stresses are implemented in Wallin and Johansson's EARSM and applied to the canonical shock-turbulence interaction. The resulting model solution is evaluated against the DNS data available in literature. The current predictions show significant improvement over existing models, for both streamwise and transverse Reynolds stresses. The jump in the turbulence quantities and post-shock evolution matches well with DNS data for Mach numbers ranging from low supersonic to the hypersonic regime.

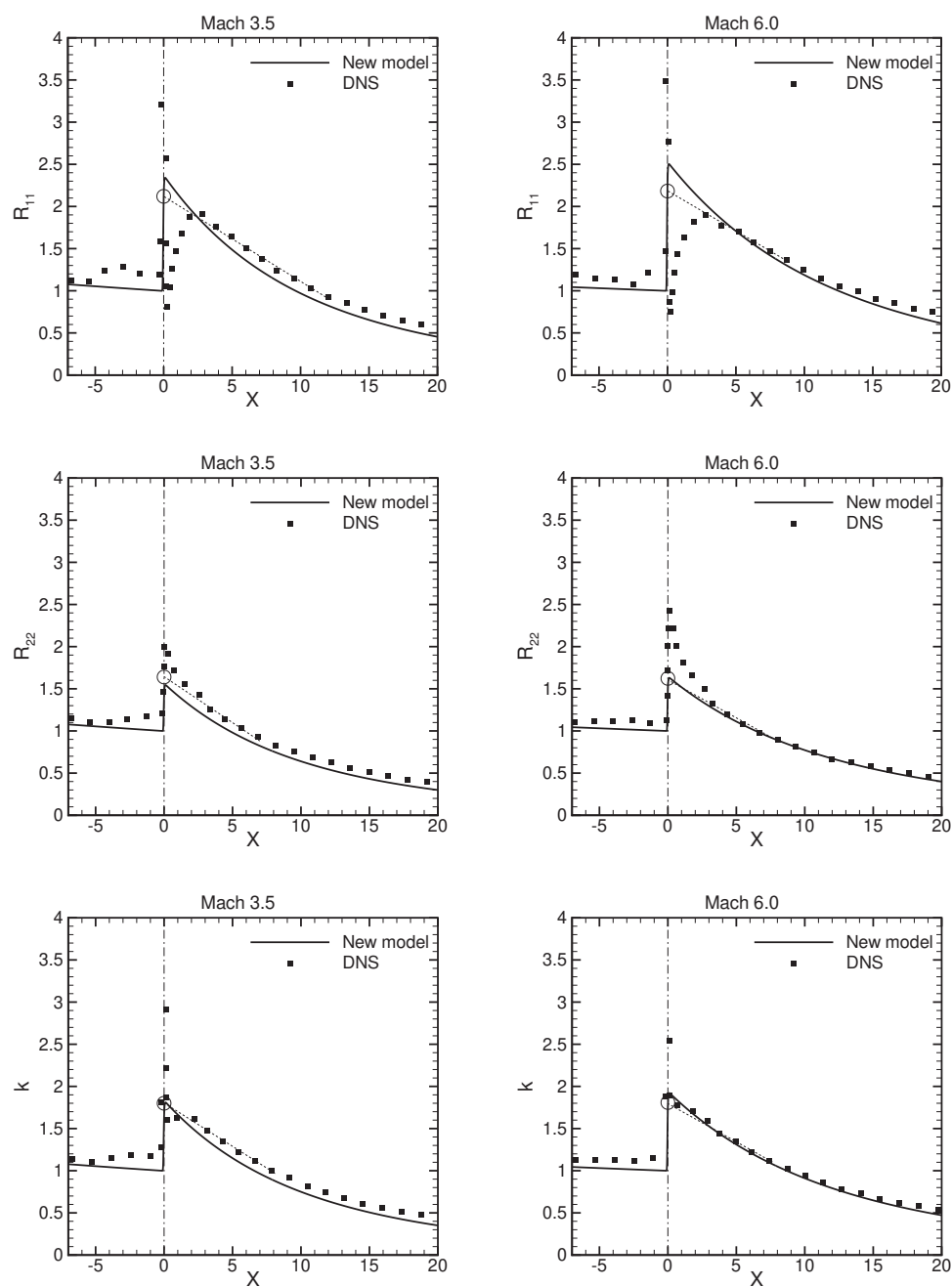


Fig. 7 Comparison of spatial variation of the normalized R_{11} , R_{22} , and TKE obtained using the new model against the DNS data at $M = 3.5$ and 6.0 .

Further, the model is able to reproduce the variation of Reynolds stress anisotropy behind the shock with increasing shock strength.

Appendix

In this appendix, we show the approximation of the solution for the term N and the coefficient β_1 . The solution for the term N with P_2 less than zero is

$$N = \frac{c'_1}{3} + 2(P_1^2 - P_2)^{1/6} \cos \left(\frac{1}{3} \arccos \left(\frac{P_1}{\sqrt{P_1^2 - P_2}} \right) \right).$$

In the above expression the value of *cosine* term is in a range of 0.86 to 0.89. The value of $\frac{c'_1}{3}$ is 0.6. Across the shock, $\mathcal{O}(\tau)$ and $\mathcal{O}(\partial \tilde{u} / \partial x)$ are about 10 respectively. By considering only the terms with high contribution, the solution for N reduces to

$$N \propto 1.8(P_1^2 - P_2)^{1/6} \implies c \left(\frac{k}{\epsilon} \right) \left\| \left(\frac{\partial u_1}{\partial x_1} \right) \right\|,$$

where

$$\begin{aligned} P_1 &= \frac{9}{20} II_S, \\ P_2 &= \left(\frac{9}{20} II_S \right)^2 - \left(\frac{9}{10} II_S \right)^3, \\ N &= 2 \times \left(\frac{9}{10} II_S \right)^{0.5} = 1.70 II_S^{0.5} \implies 1.39 \left(\frac{k}{\epsilon} \right) \left\| \left(\frac{\partial u}{\partial x} \right) \right\|. \end{aligned}$$

Computed β_1 coefficient value is

$$\beta_1 = -\frac{6}{5} \frac{1}{(1.39\tau \left\| \left(\frac{\partial u}{\partial x} \right) \right\|)}.$$

Substituting the β_1 coefficient value in the Reynolds stress tensor gives the streamwise Reynolds stress, R_{11} as,

$$\begin{aligned} R_{11} &= \frac{2}{3} \rho k + \left(0.573 \frac{1}{(\tau \left\| \left(\frac{\partial u}{\partial x} \right) \right\|)} \right) \rho k \tau \frac{\partial u}{\partial x}, \\ R_{11} &= \frac{2}{3} \rho k + 0.573 \rho k. \end{aligned}$$

The closed form solution for the TKE amplification across the shock is

$$\frac{k_d}{k_u} = \left(\frac{\widetilde{u_u}}{\widetilde{u_d}} \right)^{2/3+0.573}.$$

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