

Analysis and modeling of thermodynamic fluctuations generated by shock-turbulence interaction

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We propose simple models for the thermodynamic fluctuations in canonical shock-turbulence interaction based on the physical mechanisms observed from linear interaction analysis (LIA). The models are developed in conjunction with the two-equation $k - \epsilon$ model. The production of temperature, density, entropy and pressure variances across the shock is modeled in terms of the turbulent kinetic energy (TKE) and their corresponding decay mechanism is modeled as per the relevant physical mechanism observed in LIA. The pressure and the entropy variances are modeled with acoustic and viscous decay mechanisms, respectively. The decay mechanism of temperature variance employs a model of mixed type, which accounts for both the acoustic and the entropy mode contributions in the evolution of the temperature variance. The proposed models compare well with the direct numerical simulation (DNS) data.

I. Nomenclature

$\bar{f} \text{ \& } f'$	= Reynolds-average of variable, f and its corresponding fluctuation
$f_1 \text{ \& } f_2$	= Upstream and downstream value of variable, f
x	= shock-normal (also streamwise) direction
ρ	= density
u	= shock-normal velocity
p	= pressure
T	= temperature
R	= specific gas constant
$c_p \text{ \& } c_v$	= gas specific heat at constant pressure and constant volume
γ	= ratio of gas specific heats
a	= sound speed
M	= Mach number
k_0	= inflow turbulence peak energy wavenumber
$k \text{ \& } \epsilon$	= turbulent kinetic energy and its dissipation rate
M_t	= turbulent Mach number
Re_λ	= Reynolds number based on Taylor lengthscale, λ
ξ_t	= shock unsteadiness velocity
$\overline{f'^2}$	= statistical variance of the fluctuation in variable, f
f^*	= dimensional form of the variable, f

II. Introduction

Shock waves in high-speed compressible turbulent flows are characteristic features in the supersonic/hypersonic flow regime. Shock-turbulence interaction has implications in a variety of applications, to name a few, supersonic/hypersonic propulsion systems, inertial confinement fusion, shock wave lithotripsy, and astrophysical shock waves. In aerospace applications, the effect of shock waves on turbulence is usually studied to understand the physical mechanisms responsible for boundary layer separation, increased heat transfer and high surface pressures in the vehicle surface.

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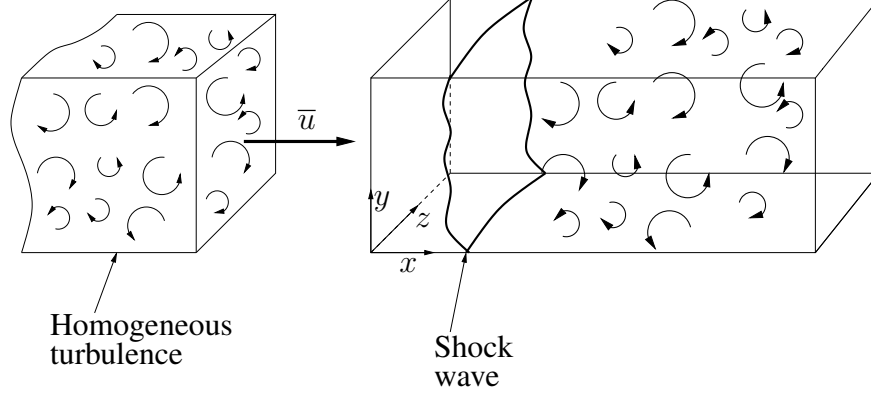


Fig. 1 Schematic of the shock-turbulence interaction where the homogeneous/isotropic turbulence is convected into the shock at a mean velocity of $\bar{u}$ in the shock-normal direction.

Appreciable changes in thermodynamic quantities such as density, pressure and temperature fluctuations are observed in compressible turbulent flows subjected to interaction with shock waves. Thermodynamic fluctuations play a vital role in turbulent mass flux, turbulent heat flux, and in the turbulent transport of energy between internal and kinetic energy components [1]. The importance of shock-induced thermodynamic fluctuations is widely known and is being explored in research areas related to Richtmyer-Meshkov instability (RMI) [2, 3], astrophysics [4, 5], jet noise [6, 7] and surface heat transfer rate [8].

We consider the interaction of homogeneous isotropic turbulence interacting with a normal shock wave as the model problem to study the shock-amplified thermodynamic fluctuations. The flow comprises of a uniform unidirectional mean flow carrying a three-dimensional turbulence towards a nominally planar normal shock as shown in Fig. 1. This canonical problem is easy to study without any additional complexities such as mean shear, streamline curvature and wall effects, etc.

The objective of the present study is to perform a detailed investigation of the thermodynamic aspects of the canonical shock-turbulence interaction and develop simple, yet robust models for the post-shock thermodynamic fluctuations. We make use of the theoretical tool called linear interaction analysis (LIA) [9] and available $k - \epsilon$ models to develop the models for the shock-amplified thermodynamic fluctuations. The models have been developed in the phenomenological sense, and are compliant with the underlying physical mechanisms. Direct numerical simulation (DNS) data from the $M_t = 0.22$ and $Re_\lambda = 40$ dataset of Larsson et al. [10] are used to validate the proposed models. The models assume that the inflow turbulence is purely vortical, which implies that there are no thermodynamic fluctuations upstream of the shock wave. The DNS data has finite thermodynamic fluctuations upstream of the shock, but are very small compared to the vortical fluctuations [11]. We further consider that this canonical problem can be represented in a one-dimensional framework, since the averages (or moments in general) are functions of only the shock-normal direction.

III. Transport equations

The governing equations for the transport of thermodynamic variances and the budget analysis of the DNS data showing the dominant terms for each of these variances, are provided in Ref. [12]. In this section, the transport equations of the thermodynamic variances are linearized (about the uniform mean flow) and the budget of each term is computed using LIA. The differential form of the linearized equations is written in the frame of reference attached to the unsteady shock [9]. The equations are then integrated across the shock to identify the contribution of each term to the amplification of the variance in the post-shock region (see Refs. [13, 14] for a detailed procedure).

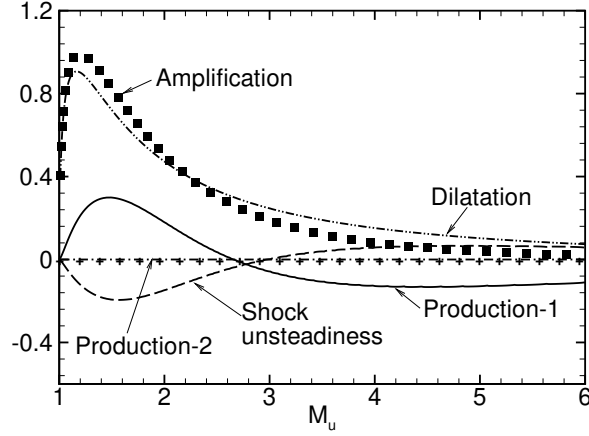


Fig. 2 Budget of Eq. 2 for different Mach numbers. The terms are shown as they appear in the equation with the plusses denoting the balance between the LHS and RHS. All of the terms are normalized by $\bar{\rho}_2^2 \bar{u}_2 (k_1/\bar{u}_1^2)$.

A. Density variance

The linearized transport equations for the density variance is obtained from the mass conservation equation by taking a moment with ρ' and neglecting the higher order terms. The linearized equation is given by,

$$\bar{u} \frac{\partial}{\partial x} \left(\frac{\overline{\rho'^2}}{2} \right) = -\overline{u' \rho'} \frac{\partial \bar{\rho}}{\partial x} - \bar{\rho'^2} \frac{\partial \bar{u}}{\partial x} + \overline{\rho' \xi_t} \frac{\partial \bar{\rho}}{\partial x} - \bar{\rho} \left(\overline{\rho' \frac{\partial u'}{\partial x}} \right), \quad (1)$$

where the first and the second terms on the RHS are production terms due to mean gradients of density and velocity, respectively. The third term accounts for the damping in production due to the unsteady shock and the last term is the correlation between density and dilatation fluctuations. From the linearized Rankine-Hugoniot (RH) relation for conservation of mass, the change in density variance across the shock is written as,

$$\underbrace{\bar{u}_2 \frac{(\overline{\rho_2'^2} - \overline{\rho_1'^2})}{2}}_{\text{Amplification}} = - \underbrace{\overline{\rho_m' u_2'} (\bar{\rho}_2 - \bar{\rho}_1)}_{\text{Production-1}} - \underbrace{\overline{\rho_1' \rho_m'} (\bar{u}_2 - \bar{u}_1)}_{\text{Production-2}} + \underbrace{\overline{\rho_m' \xi_t} (\bar{\rho}_2 - \bar{\rho}_1)}_{\text{Shock unsteadiness}} - \underbrace{\bar{\rho}_1 \overline{\rho_m' (u_2' - u_1')}}_{\text{Dilatation}}, \quad (2)$$

where $\rho_m' = (\rho_1' + \rho_2')/2$. Equation 2 can be interpreted as the integrated form of Eq. 1 across the shock. The term *Amplification* describes the change in density variance across the shock, *Production-1* is the production due to mean density gradients, *Production-2* is the production due to mean velocity gradients, *Shock unsteadiness* is the damping term due to the unsteady shock, and *Dilatation* corresponds to the density-dilatation correlation term.

Figure 2 shows the budget of Eq. 2 against different Mach numbers for the case of purely vortical inflow turbulence, where the terms are normalized by $\bar{\rho}_2^2 \bar{u}_2 (k_1/\bar{u}_1^2)$. The integrated budgets across the shock show that the *Dilatation* term to be the dominant term matching the *Amplification* term. The effect of the *Shock unsteadiness* term is found to damp the *Production-1* term. The *Production-2* term is identically zero due to assumption of purely vortical turbulence upstream of the shock. The total density variance immediately behind the shock (or nearfield density variance) tend towards zero at high Mach numbers, which is due to the balance between the acoustic and the entropy modes. The density variance away from the shock (or farfield density variance), however, is finite due to the entropy mode being larger than the acoustic mode (see Ref. [12] for details). For the case of upstream turbulence comprising of both vorticity and entropy fluctuations, the nearfield density variance is found to be finite for the high Mach number interactions and strongly depends on the correlation between the vorticity and the entropy modes [9].

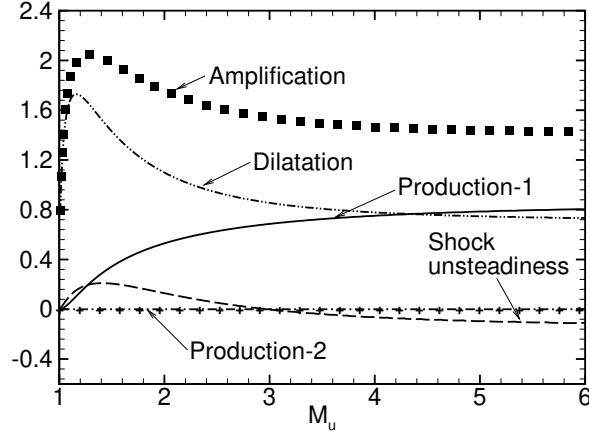


Fig. 3 Budget of Eq. 4 for different Mach numbers. The terms are shown as they appear in the equation with the plusses denoting the balance between the LHS and RHS. All of the terms are normalized by $\bar{p}_2^2 (k_1/\bar{u}_1^2)$.

B. Pressure variance

The linearized transport equations for the pressure variance is also obtained in a similar way as that of the density variance. Taking a moment of the streamwise momentum conservation equation with p' and neglecting the higher order terms yields the linearized equation for the pressure variance,

$$\frac{\partial}{\partial x} \left(\frac{\overline{p'^2}}{2} \right) = -\bar{\rho} \overline{p' u'} \frac{\partial \bar{u}}{\partial x} - \bar{u} \overline{p' \rho'} \frac{\partial \bar{u}}{\partial x} + \bar{\rho} \overline{p' \xi_t} \frac{\partial \bar{u}}{\partial x} - \bar{\rho} \bar{u} \overline{p' \frac{\partial u'}{\partial x}}, \quad (3)$$

which is similar to Eq. 1 with the first and the second terms in the RHS being the production terms, the third term is the shock unsteadiness term, which damps the production due to mean velocity gradients and the last term is the correlation between the pressure and the dilatation fluctuations. The change in pressure variance across the shock is now obtained from the linearized RH relation for conservation of streamwise momentum,

$$\underbrace{\frac{\bar{p}_2'^2 - \bar{p}_1'^2}{2}}_{\text{Amplification}} = -\underbrace{\bar{\rho}_1 \overline{p'_m u'_1} (\bar{u}_2 - \bar{u}_1)}_{\text{Production-1}} - \underbrace{\bar{u}_1 \overline{p'_m \rho'_1} (\bar{u}_2 - \bar{u}_1)}_{\text{Production-2}} + \underbrace{\bar{\rho}_1 \overline{p'_m \xi_t} (\bar{u}_2 - \bar{u}_1)}_{\text{Shock unsteadiness}} - \underbrace{\bar{\rho}_1 \bar{u}_1 \overline{p'_m (u'_2 - u'_1)}}_{\text{Dilatation}}, \quad (4)$$

where $p'_m = (p'_1 + p'_2)/2$. The above equation is analogous to Eq. 2 and can be taken as the integrated form of Eq. 3, with the first two terms representing the production of pressure variance, the third term represents the shock unsteadiness effect and the last term denotes the pressure-dilatation correlation.

Figure 3 shows the budget of Eq. 4 for different Mach numbers, where the terms are normalized by $\bar{p}_2^2 (k_1/\bar{u}_1^2)$. The integrated budgets across the shock show that the *Production-1* and the *Dilatation* term to be the dominant terms whereas, the effect of *Shock unsteadiness* is negligible except for the low Mach number cases. The *Production-2* term which comprises of the correlation of pressure fluctuations with upstream density fluctuations is identically zero due to the consideration of purely vortical turbulence upstream of the shock. This term, however, would have significant contribution to the change in pressure variance if the upstream turbulence is of a mixed type such as vortical-entropy turbulence as considered in Refs. [14, 15].

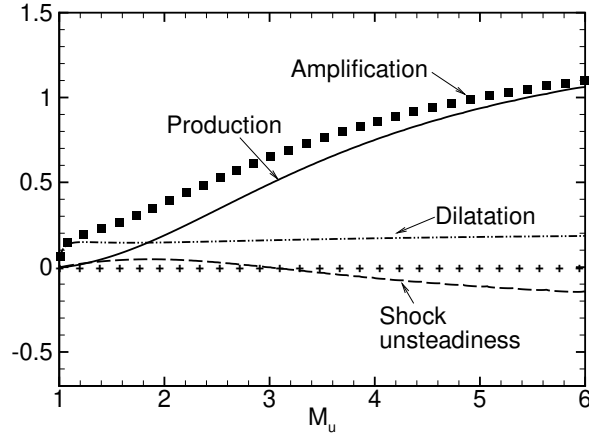


Fig. 4 Budget of Eq. 6 for different Mach numbers. The terms are shown as they appear in the equation with the plusses denoting the balance between the LHS and RHS. All of the terms are normalized by $c_p \bar{T}_2^2 (k_1/\bar{u}_1^2)$.

C. Temperature variance

The conservation of total energy in its linearized form is used to obtain the linearized transport equation for the temperature variance in the unsteady shock frame of reference, where a moment with T' is taken and the higher order terms are neglected. The linearized equation is,

$$c_p \frac{\partial}{\partial x} \left(\frac{\overline{T'^2}}{2} \right) = -\overline{u'T'} \frac{\partial \bar{u}}{\partial x} + \overline{T'\xi_t} \frac{\partial \bar{u}}{\partial x} - \overline{\bar{u}T'} \frac{\partial u'}{\partial x}, \quad (5)$$

where the first term on the RHS is the production term due to the gradient of mean velocity. The second term is the shock unsteadiness term and the last term is the correlation between temperature and dilatation fluctuations. From the linearized RH relation for enthalpy conservation across a normal shock, the change in temperature variance is written as,

$$\underbrace{c_p \frac{(\overline{T_2'^2} - \overline{T_1'^2})}{2}}_{\text{Amplification}} = \underbrace{-\overline{u_1' T_m'}}_{\text{Production}} (\bar{u}_2 - \bar{u}_1) + \underbrace{\overline{T_m' \xi_t}}_{\text{Shock unsteadiness}} (\bar{u}_2 - \bar{u}_1) - \underbrace{\overline{\bar{u}_2 T_m' (u_2' - u_1')}}_{\text{Dilatation}}, \quad (6)$$

where $T_m' = (T_1' + T_2')/2$. Equation 6 is the integrated form of Eq. 5 across the shock, where the first term on the RHS is the production due to mean velocity gradient (*Production*), followed by the damping of the *Shock unsteadiness* term and the temperature-dilatation correlation term (*Dilatation*).

Figure 4 shows the budget of Eq. 6 for different Mach numbers where the terms are normalized by $c_p \bar{T}_2^2 (k_1/\bar{u}_1^2)$. The integrated budget for the temperature variance across the shock show that the *Production* term to be the dominant term, contributing to the entirety of the amplification of temperature variance (*Amplification*) across the shock. The *Dilatation* term albeit small, is finite and is responsible for the amplification of temperature variance for the low Mach number cases. The *Shock unsteadiness* term is found to be negligible except for the high Mach number cases, where it balances out the contribution from the *Dilatation* term.

IV. Model development

The predictive models for temperature, density, pressure and entropy fluctuations are developed from the linearized equations in this section. The procedure used in the development of the models is analogous to the procedure followed in the works of Sinha et al. [13] and Quadros & Sinha [16]. We make use of the understanding of the underlying dynamics from earlier analysis on the thermodynamic fluctuations [12] and on the turbulent energy flux [8, 17] for the model development. The following points consolidate the key messages from the previous studies:

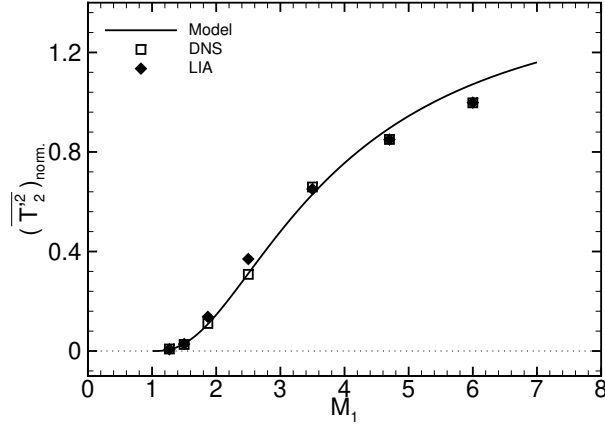


Fig. 5 Variation of normalized temperature variance against Mach number. Normalization is as mentioned in Eq. 10. LIA and DNS values (taken by extrapolating to the mean shock centre) are shown by a diamond and square symbols, respectively. The predictions from Eq. 10 is shown by a solid line.

- Thermodynamic variances obtained from LIA match well with the DNS data, and thus LIA predictions can be used to model the underlying physics.
- Reynolds and Favre averaged quantities were found to be approximately the same, i.e., $\overline{f} \approx \widetilde{f}$ and $\overline{f'^2} \approx \widetilde{f'^2}$, where the tilde and double-prime correspond to Favre averages and its fluctuations, respectively.
- Density ($\overline{\rho'^2}$), pressure ($\overline{p'^2}$) and temperature ($\overline{T'^2}$) variances show a large amplification at the shock followed by a rapid decay. This rapid decay is mostly due to acoustic mechanisms.
- Entropy variance ($\overline{s'^2}$) also show a large amplification at the shock, and a rather gradual decay compared to the other thermodynamic quantities. This slower decay is mostly due to viscous mechanisms.
- Decomposition of the post-shock thermodynamic variances in terms of their Kovásznyai modes [18] (acoustic, entropy and vorticity modes) showed that the entropy-entropy correlation to be significant in the farfield ($x \rightarrow \infty$) and the acoustic-acoustic correlation to be dominant in the nearfield ($x = 0^+$). Here, the shock is located at $x = 0$.
- At high Mach numbers, the effect of the acoustic mode in the variances is found to be small compared to the entropy mode in the thermodynamic fluctuations.

A. Temperature variance

The amplification of temperature fluctuations behind the normal shock is modeled using the *Production* term as,

$$\frac{\partial \overline{T'^2}}{\partial x} \approx -2 \frac{1}{c_p} \overline{u'T'} \frac{\partial \bar{u}}{\partial x}, \quad (7)$$

where the unknown turbulent temperature flux correlation is closed using the model (see Appendix for more details),

$$\overline{u'T'} = \underbrace{\left[\frac{4}{9}(1 - b_1) \right] \left(\frac{3}{2b_1 + 1} \right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2} \right)^{-\frac{1}{3}(2b_1 + 1)} \right]}_{C_{uT}} \frac{k \bar{u}}{c_p}, \quad (8)$$

with $b_1 = 0.4 + 0.6 e^{2(1-M_1)}$. The modeled equation for the amplification of temperature variance across the shock reads as,

$$\frac{\partial \overline{T'^2}}{\partial x} \approx -2 \frac{C_{uT} k \bar{u}}{c_p^2} \frac{\partial \bar{u}}{\partial x}, \quad (9)$$

which depends on the accurate modeling of k . The shock unsteadiness $k - \epsilon$ model of Sinha et al. [13] (SU $k - \epsilon$ model) is used for modeling the TKE. The above equation can be integrated across the shock to yield a closed form

solution for the normalized temperature variance as,

$$\left(\overline{T'^2}\right)_{norm.} = \frac{\overline{T'^2}}{\overline{T}_2^2 \left(\frac{k_1}{\bar{u}_1^2}\right)} \approx 2 C_{uT} \left\{ \left(\frac{3}{2(b_1+2)}\right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2}\right)^{-\frac{2}{3}(2+b_1)}\right] \right\} (\gamma-1)^2 \frac{M_1^4}{(\overline{T}_2/\overline{T}_1)^2}. \quad (10)$$

The assumption of purely vortical turbulence upstream of the shock is also considered, which results in $\overline{T'^2} = 0$. The normalized temperature variance is a function of the upstream Mach number and the ratio of specific heats only, which makes it suitable to be compared with the LIA data.

Figure 5 shows the closed-form solution of the normalized temperature variance (Eq. 10) in comparison to the LIA and DNS results against a range of Mach numbers. The DNS dataset used for comparison is the $M_t = 0.22$ and $Re_\lambda = 40$ dataset of Larsson et al. [10, 11] which spans from $M_1 = 1.27$ to 6. DNS and LIA show a good match for the temperature variance till the location $k_0 x \approx 1$. Beyond $k_0 x = 1$, LIA predicts constant values whereas, DNS shows further decay towards zero, though gradually, due to the viscous effects. The closed-form solution of the model given in Eq. 10 predicts values that are closer to the LIA and the DNS results, which were extrapolated to the mean shock centre from $k_0 x = 1$. This model along with a modeled decay mechanism is shown in the next section (Sec. V) to have an excellent match with the DNS profile.

B. Density variance

The amplification of density variance can also be modeled in a similar fashion as that of the temperature variance shown above. The linearized equation for the density variance is reduced to,

$$\frac{\partial \overline{\rho'^2}}{\partial x} \approx -2 \frac{1}{\bar{u}} \overline{u' \rho'} \frac{\partial \bar{\rho}}{\partial x}, \quad (11)$$

where we have to model the unknown $\overline{\rho' u'}$ correlation (turbulent mass flux) to provide closure. We attempt to model the farfield density variance in similar lines as that of the model development of farfield k in Ref. [13]. LIA suggests that the farfield acoustic mode becomes very small compared to the entropy mode at high Mach numbers. We make use of this understanding to model the turbulent mass flux in terms of the modeled $\overline{u' T'}$ through the following relations,

$$p' = 0 \quad \Rightarrow \quad \frac{\rho'}{\bar{\rho}} = -\frac{T'}{\bar{T}}, \quad (12)$$

$$\overline{\rho' u'} = -\frac{\bar{\rho}}{\bar{T}} \overline{u' T'} = -\frac{\bar{\rho}}{\bar{T}} C_{uT} \frac{k \bar{u}}{c_p}, \quad (13)$$

where C_{uT} is the expression given in Eq. 8. The modeled production of density variance then becomes,

$$\frac{\partial \overline{\rho'^2}}{\partial x} \approx -2 \frac{\bar{\rho}^2}{c_p \bar{T}} C_{uT} \frac{k}{\bar{u}} \frac{\partial \bar{u}}{\partial x}, \quad (14)$$

where the conservation of averaged mass,

$$\frac{\partial \bar{\rho}}{\partial x} = -\frac{\bar{\rho}}{\bar{u}} \frac{\partial \bar{u}}{\partial x}, \quad (15)$$

is used to replace the mean density gradient in terms of the mean velocity gradient.

The following relations are used to rewrite Eq. 14 to be in terms of the mean velocity, $\bar{u}$ only,

$$\frac{\bar{\rho}_2}{\bar{\rho}_1} = \frac{\bar{u}_1}{\bar{u}_2}, \quad c_p \bar{T}_2 + \frac{\bar{u}_2^2}{2} = c_p \bar{T}_1 + \frac{\bar{u}_1^2}{2}, \quad \frac{k_2}{k_1} = \left(\frac{\bar{u}_1}{\bar{u}_2}\right)^{\frac{2}{3}(1-b_1)}, \quad (16)$$

which upon integration (across the shock) yields,

$$\begin{aligned} \left(\overline{\rho'^2}\right)_{norm.} = \frac{\overline{\rho'^2}}{\bar{\rho}_2^2 \left(\frac{k_1}{\bar{u}_1^2}\right)} \approx & 4 \frac{C_{uT}}{(\bar{\rho}_2/\bar{\rho}_1)^2} \left[\frac{\left(\frac{\gamma-1}{2}\right) M_1^2}{1 + \left(\frac{\gamma-1}{2}\right) M_1^2} \right] \left\{ \left(\frac{3}{2(b_1-4)}\right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2}\right)^{-\frac{2}{3}(b_1-4)}\right] \right. \\ & \left. + \left[\frac{\left(\frac{\gamma-1}{2}\right) M_1^2}{1 + \left(\frac{\gamma-1}{2}\right) M_1^2} \right] \left(\frac{3}{2(b_1-1)}\right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2}\right)^{-\frac{2}{3}(b_1-1)}\right] \right\}, \end{aligned} \quad (17)$$

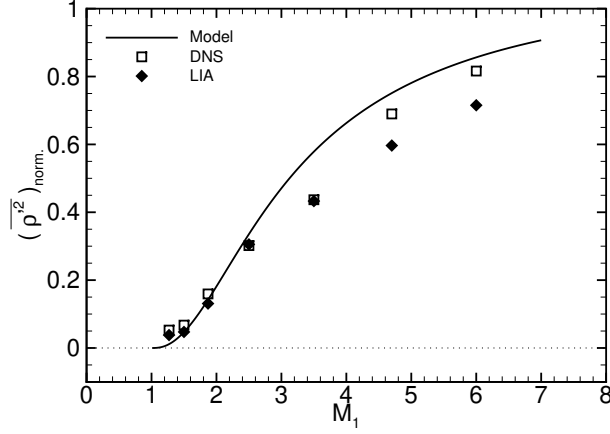


Fig. 6 Variation of normalized density variance against Mach number. Normalization is as mentioned in Eq. 17. LIA and DNS values (extrapolated to the mean shock centre) are shown by diamond and square symbols, respectively. The values from the closed-form solution given in Eq. 17 is shown by a solid line.

which is an approximation of the exact closed form solution, since the Taylor series expansion of $(c_p \bar{T})^{-1}$ in Eq. 14 is limited up to first order terms only. The upstream density variance, $\overline{\rho'^2} = 0$, due to the assumption of purely vortical turbulence upstream of the shock.

Figure 6 compares the closed-form solution of the normalized density variance given in Eq. 17 with the LIA and the DNS results for a range of Mach numbers. The model overpredicts, albeit slightly, in comparison to both the DNS and the LIA data, whose values were taken by extrapolating to the mean shock centre from the $k_0 x = 1$ location. For the high Mach number cases, both LIA and DNS show a nonmonotonic variation in density variance with a post-shock peak located approximately at the location $k_0 x = 1$. LIA predicts constant value of density variance beyond $k_0 x = 3$ location after a short length of adjustment whereas, DNS shows a gradual decay towards zero similar to the temperature variance profile. The negative correlation between the acoustic and the entropy modes is found to be the reason for this nonmonotonic behavior in density variance whereas, the correlation was found to be positive for the temperature variance giving it a monotonic profile (see Ref. [12]).

C. Entropy variance

Entropy is defined as,

$$s = c_v \log \left(\frac{p}{\rho^\gamma} \right) = c_v \log \left(\frac{RT}{\rho^{(\gamma-1)}} \right), \quad (18)$$

which is a derived quantity based on the other thermodynamic quantities (see Ref. [19]). The procedure used in the derivation of the linearized transport of density, pressure and temperature variances is not directly applicable for the entropy variance, since entropy is not conserved across the shock. We make use of the transport equation of entropy variance [12] which is valid in the post-shock region to develop the model, assuming that the flowfield has been altered by the shock wave. The transport equation for the entropy variance with only the production term reads as,

$$\frac{\partial \overline{s'^2}}{\partial x} \approx -2 \frac{1}{\bar{u}} \overline{s' u'} \frac{\partial \bar{s}}{\partial x}, \quad (19)$$

where we have considered only the production term due to the mean entropy gradient across the shock as the source term (i.e., neglected source terms due to other mean gradients). We model the entropy flux correlation in terms of the modeled temperature flux correlation,

$$\frac{s'}{c_p} = -\frac{\rho'}{\bar{\rho}} = \frac{T'}{\bar{T}} \Rightarrow \overline{s' u'} = \frac{c_p}{\bar{T}} \overline{u' T'}, \quad (20)$$

where we have used the fact that $p'/\bar{p} \ll s'/c_p$ for the high Mach number cases considered in this study. We bring in the effect of the shock wave into Eq. 19 via the modeled $\overline{u' T'}$ term. The mean entropy gradient in Eq. 19 is written in

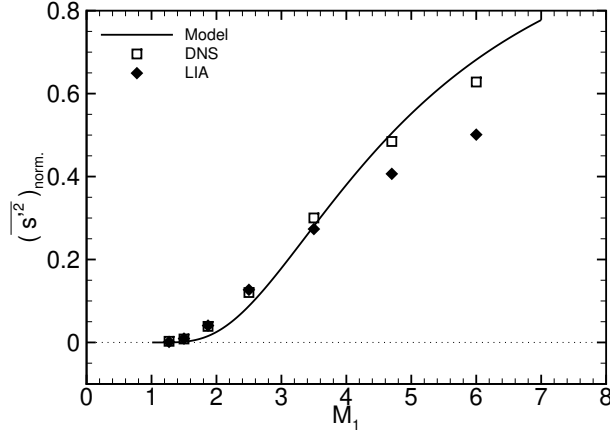


Fig. 7 Variation of normalized entropy variance against Mach number. Normalization is as mentioned in Eq. 23. DNS values are taken by extrapolating to the mean shock centre, which are shown by square symbols, and the LIA values are shown by diamond symbols. The closed-form solution given in Eq. 23 is shown by a solid line.

terms of the mean velocity gradient by averaging Eq. 18 and then computing its gradient,

$$\frac{\partial \bar{s}}{\partial x} = c_v \left[\frac{\gamma - 1}{\bar{u}} - \frac{\bar{u}}{c_p \bar{T}} \right] \frac{\partial \bar{u}}{\partial x}. \quad (21)$$

The modeled entropy variance equation is now written in terms of turbulent kinetic energy and mean velocity gradients as,

$$\frac{1}{c_p^2} \frac{\partial \bar{s}^2}{\partial x} = -2 \frac{C_{uT}}{\gamma} \frac{k}{c_p \bar{T}} \left[(\gamma - 1) - \frac{\bar{u}^2}{c_p \bar{T}} \right] \frac{1}{\bar{u}} \frac{\partial \bar{u}}{\partial x}, \quad (22)$$

where C_{uT} is the same model coefficient given in Eq. 8. Integration of Eq. 22 across the shock results in,

$$\begin{aligned} \left(\bar{s}^2 \right)_{norm.} = \frac{\bar{s}_2^2}{c_p^2 \left(\frac{k_1}{\bar{u}_1^2} \right)} &\approx 4 \frac{C_{uT}}{\gamma} \left[\frac{\left(\frac{\gamma-1}{2} \right) M_1^2}{1 + \left(\frac{\gamma-1}{2} \right) M_1^2} \right] \times \\ &\left\{ (\gamma - 1) \left(\frac{3}{2(b_1 - 1)} \right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2} \right)^{-\frac{2}{3}(b_1-1)} \right] + (\gamma - 3) \left[\frac{\left(\frac{\gamma-1}{2} \right) M_1^2}{1 + \left(\frac{\gamma-1}{2} \right) M_1^2} \right] \left(\frac{3}{2(b_1 + 2)} \right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2} \right)^{-\frac{2}{3}(b_1+2)} \right] \right. \\ &\left. - 4 \left[\frac{\left(\frac{\gamma-1}{2} \right) M_1^2}{1 + \left(\frac{\gamma-1}{2} \right) M_1^2} \right]^2 \left(\frac{3}{2(b_1 + 5)} \right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2} \right)^{-\frac{2}{3}(b_1+5)} \right] - 2 \left[\frac{\left(\frac{\gamma-1}{2} \right) M_1^2}{1 + \left(\frac{\gamma-1}{2} \right) M_1^2} \right]^3 \left(\frac{3}{2(b_1 + 8)} \right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2} \right)^{-\frac{2}{3}(b_1+8)} \right] \right\}, \end{aligned} \quad (23)$$

which is an approximation to the exact closure of the equation due to the truncation of the Taylor series expansion of $(c_p \bar{T})^{-2}$ to the first order terms only.

The production of the normalized entropy variance across the shock is given by the closed-form solution of Eq. 22. The variation of Eq. 23 is shown in Fig. 7 along with the LIA and the DNS results. DNS predicts large amplification of entropy variance at the shock followed by a gradual decay. LIA, on the other hand, predicts constant entropy variance after the shock and throughout the downstream region. The model compares well with the DNS and the LIA data, although it slightly underpredicts for the low Mach number cases and overpredicts for the high Mach numbers. The DNS values are once again taken by extrapolating to the mean shock centre from the $k_0 x = 1$ location.

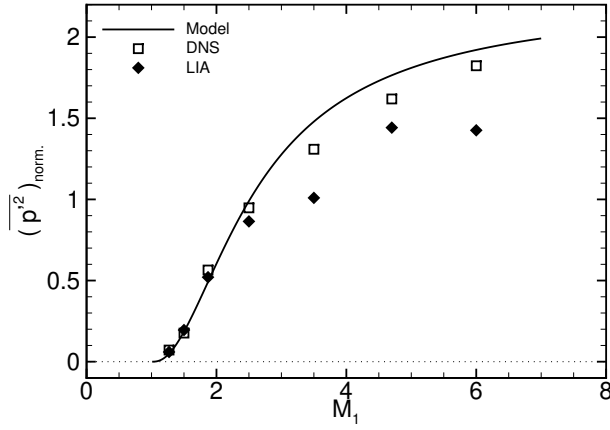


Fig. 8 Variation of normalized pressure variance against Mach number. Normalization is as mentioned in Eq. 27. LIA and DNS values are taken by extrapolating to the centre of the mean shock [17]. The LIA and DNS results are shown by diamond and square symbols, respectively. The closed-form solution given in Eq. 27 is shown by a solid line.

D. Pressure variance

The model equation for the production of pressure variance is given by,

$$\frac{\partial \overline{p'^2}}{\partial x} \approx -2 \bar{\rho} \overline{p'u'} \frac{\partial \bar{u}}{\partial x}, \quad (24)$$

where only the production term in Eq. 3 is considered to be the source term.

The production of the density and entropy variances were obtained by considering that there are negligible pressure fluctuations for the high Mach number interactions. We attempt to develop the model for pressure variance with a different approach whereby, we use isentropic relations to relate the pressure flux correlation $\overline{p'u'}$ and the temperature flux correlation $\overline{T'u'}$ as,

$$\frac{p'}{\bar{p}} = \left(\frac{\gamma}{\gamma - 1} \right) \frac{T'}{\bar{T}} \Rightarrow \overline{p'u'} = \left(\frac{\gamma}{\gamma - 1} \right) \frac{\bar{p}}{\bar{T}} \overline{u'T'}. \quad (25)$$

Upon substitution of Eq. 25 in Eq. 24 we get,

$$\frac{\partial \overline{p'^2}}{\partial x} \approx -2 \bar{p}^2 C_{uT} k \bar{u} \frac{\partial \bar{u}}{\partial x}, \quad (26)$$

where Eq. 8 is used to write the temperature flux correlation in terms of the turbulent kinetic energy. Integration of Eq. 26 yields,

$$\left(\overline{p'^2} \right)_{norm.} = \frac{\overline{p'^2}}{\bar{p}^2 \left(\frac{k_1}{\bar{u}_1^2} \right)} \approx 2 C_{uT} \left(\frac{3}{2(b_1 - 1)} \right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2} \right)^{-\frac{2}{3}(b_1 - 1)} \right] \gamma^2 \frac{M_1^4}{(\bar{p}_2/\bar{p}_1)^2}, \quad (27)$$

where the normalized pressure variance is found to approach an asymptotic value at high Mach numbers similar to the temperature variance.

Figure 8 shows the closed-form solution given in Eq. 27 along with the results from LIA and DNS for a range of Mach numbers. The model predicts amplifications slightly larger than the DNS values, which were taken by extrapolating to the mean shock centre from the location, $k_{0x} = 1$. The LIA values are also computed in a similar fashion by extrapolating to the shock centre. Unlike the DNS values which were increasing as the Mach number is increased, the LIA values were found to asymptote around the value of 1.5 for the high Mach number cases. This results in the model to be largely overpredicting with respect to the LIA results. Nonetheless, the model shows excellent agreement with the available DNS data.

V. Model predictions

The averages (or moments) for this case of canonical shock-turbulence interaction are only dependent on the streamwise direction. This enables us to solve the transport equations for the thermodynamic variances given in Eqs. 9, 14, 22 and 26 in a one-dimensional framework. Since the equations are solved in the one-dimensional framework, the partial differential operator (∂) and the ordinary differential operator (d) mean the same, and are used interchangeably. The equations are integrated in space using the classical 4th order accurate Runge-Kutta method. The mean profile with the shock located at $k_0 x = 0$ is specified as follows,

$$\bar{u}(x) = \bar{u}_1 + (\bar{u}_2 - \bar{u}_1) \frac{1}{2} (1 + \tanh(x)), \quad (28)$$

$$\bar{\rho}(x) = \bar{\rho}_1 \frac{\bar{u}_1}{\bar{u}(x)}, \quad (29)$$

$$\bar{T}(x) = \bar{T}_1 + \frac{1}{2c_p} (\bar{u}_1^2 - \bar{u}(x)^2), \quad (30)$$

$$\frac{d\bar{u}(x)}{dx} = \frac{\bar{u}_2 - \bar{u}_1}{\Delta x} \frac{1}{2} (1 - \tanh^2(x)), \quad (31)$$

$$\frac{d\bar{T}(x)}{dx} = -\frac{1}{c_p} \bar{u}(x) \frac{d\bar{u}(x)}{dx}, \quad (32)$$

where Δx is the grid spacing in the one-dimensional grid.

The transport equations for the thermodynamic variances depend on the accurate modeling of TKE, k and its dissipation rate, ϵ . The model equations for k and ϵ from Refs. [13, 20] have proven to predict well for shock-dominated problems. The k and ϵ equations are also integrated simultaneously along with the equations for the thermodynamic variances. The transport equations for the thermodynamic variances account only for the production of the thermodynamic variances, and their decay needs additional modeling. The spatial decay of the thermodynamic variances are modeled in a phenomenological sense, following the work of Ref. [16]. From LIA and DNS profiles of thermodynamic variances [12], the following points summarize the decay mechanisms for the thermodynamic variances:

- Temperature and density variances show acoustic decay till $k_0 x \approx 1 - 2$ and viscous decay beyond that location
- Pressure variance has only the acoustic decay mechanism
- Entropy variance has only the viscous decay mechanism

We model the decay profiles of the thermodynamic variances with the physical insights obtained from the earlier studies. The pressure and the entropy variance equations are implemented with an acoustic and a viscous decay mechanism, respectively. Temperature variance is modeled with a mixed type of decay profile with the acoustic mode being dominant behind the shock upto a certain distance and the entropy mode after that location.

The production of density variance was modeled by considering the farfield variation of the density variance. The density variance is thus, modeled with only a viscous decay mechanism since the acoustic decay effects are minimal in the farfield. The complete equations with the production and the decay terms for each of the thermodynamic variances along with the models for k and ϵ are given below,

$$\frac{\partial}{\partial x} \begin{Bmatrix} \bar{\rho} \bar{u} k \\ \bar{\rho} \bar{u} \epsilon \\ \bar{T}^2 \\ \bar{\rho}^2 \\ \bar{s}^2 \\ \bar{p}^2 \end{Bmatrix} = \begin{Bmatrix} -\frac{2}{3} \bar{\rho} k \frac{\partial \bar{u}}{\partial x} (1 - b_1') - \bar{\rho} \epsilon \\ -\frac{2}{3} c_{\epsilon 1} \bar{\rho} \epsilon \frac{\partial \bar{u}}{\partial x} - c_{\epsilon 2} \bar{\rho} \frac{\epsilon^2}{k} \\ -2 C_{uT} \frac{k \bar{u}}{c_p^2} \frac{\partial \bar{u}}{\partial x} - \Phi \\ -2 \frac{\bar{\rho}^2}{c_p \bar{T}} C_{uT} \frac{k}{\bar{u}} \frac{\partial \bar{u}}{\partial x} - C_{\rho}^{vd} \frac{1}{\bar{u}} \frac{\epsilon}{k} \bar{\rho}^2 \\ -2 C_s^P \frac{C_{uT}}{\gamma} \frac{k}{c_p \bar{T}} \left[(\gamma - 1) - \frac{\bar{u}^2}{c_p \bar{T}} \right] \frac{1}{\bar{u}} \frac{\partial \bar{u}}{\partial x} - C_s^{vd} \frac{1}{\bar{u}} \frac{\epsilon}{k} \bar{s}^2 \\ -2 \bar{\rho}^2 C_{uT} k \bar{u} \frac{\partial \bar{u}}{\partial x} - C_p^{ad} \left(\frac{\bar{p}^2 - \bar{p}^{2, LIA}}{L_{\epsilon}} \right) \end{Bmatrix}, \quad (33)$$

where the modeling parameters are defined as,

$$b_1 = 0.4 + 0.6 e^{2(1-M_1)}, \quad b'_1 = 0.4 (1 - e^{(1-M_1)}), \quad (34)$$

$$C_{uT} = \left[\frac{4}{9} (1 - b_1) \right] \left(\frac{3}{2(b_1 + 1)} \right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2} \right)^{-\frac{2}{3}(b_1+1)} \right], \quad (35)$$

$$\Phi = \left(C_T^{ad} \frac{\overline{T'^2}}{L_\epsilon} \right) + \left(C_T^{vd} \frac{1}{\bar{u}} \frac{\epsilon}{k} \overline{T'^2} - C_T^{ad} \frac{\overline{T'^2}}{L_\epsilon} \right) \frac{1}{2} (1 + \tanh(x - \alpha \lambda_0)), \quad \lambda_0 \sim \frac{1}{k_0}, \quad \alpha = \frac{1}{2} \text{ to } \frac{3}{2}, \quad (36)$$

$$c_{\epsilon 1} = 1 + 0.21 M_1, \quad c_{\epsilon 2} = 1.2, \quad (37)$$

$$C_T^{ad} = 0.95 + 9.25 e^{(1-M_1)}, \quad C_T^{vd} = 0.75 + 3.5 e^{(1-M_1)}, \quad C_\rho^{vd} = 0.6 + 2.75 e^{(1-M_1)}, \quad (38)$$

$$C_p^{ad} = 2 + 4.5 e^{(1-M_1)}, \quad C_s^P = 0.15 + 2.5 e^{(1-M_1)}, \quad C_s^{vd} = 0.75 + 3.7 e^{(1-M_1)}. \quad (39)$$

where C_f^{ad} and C_f^{vd} are the acoustic and viscous decay model coefficient for the quantity f , respectively. Here, $k_0 = 4$ is the peak energy wavenumber used in the upstream turbulence energy spectrum, and λ_0 is the corresponding wavelength.

The term $\overline{p'^2}_{farfield}^{LIA}$ is the farfield value of pressure variance obtained from LIA. The source terms in the equations for the thermodynamic variances are active only in the downstream region to be compliant with the purely vortical turbulence assumption. The source terms for k and ϵ are active in the upstream as well as the downstream region.

The decay for the temperature variance is modeled to have a mixed decay profile as shown in Eq. 36, where the hyperbolic tangent function controls the transition from the acoustic decay to the viscous decay. The transition location is identified using LIA as the location where the entropy modes become dominant over the acoustic modes.

Similar to the decay model for the turbulent energy flux proposed in Ref. [16], we also model the acoustic decay mechanism in the form of $\overline{f'^2}/L_\epsilon$, where the dissipation lengthscale is given by $L_\epsilon = k^{(3/2)}/\epsilon$. The viscous decay mechanism is similar to the turbulent kinetic energy dissipation, and is in the form of $(\epsilon/k) \overline{f'^2}$. These forms of the acoustic decay and the viscous decay are used for the other variances also. It is shown in Fig. 9 that this type of mixed decay profile with the transition location set at $k_0 x = 4/3$, matches very well with the spatial variation of temperature variance observed in the DNS.

In the previous section, it was shown that the model equation for the entropy variance was developed in the phenomenological sense from its transport equation applicable to the downstream region. The modeled entropy variance varies in the $O(M^2)$ when integrated numerically, and thus requires an additional production coefficient, C_s^P to control the values to be comparable to the post-shock predictions of LIA.

The equations are integrated in their non-dimensional form where the quantities are non-dimensionalized as shown below,

$$u = \frac{u^*}{a_1^*}, \quad T = \frac{T^*}{T_1^*(\gamma - 1)}, \quad \rho = \frac{\rho^*}{\rho_1^*}, \quad p = \frac{p^*}{p_1^* \gamma}, \quad s = \frac{s^*}{c_p^*} \quad (40)$$

$$R = \frac{R^*}{c_p^*} = \frac{\gamma - 1}{\gamma}, \quad c_p = \frac{c_p^*}{c_p^*} = 1, \quad c_v = \frac{c_v^*}{c_p^*} = \frac{1}{\gamma}, \quad x = k_0 x^*, \quad (41)$$

with the superscript $*$ denoting dimensional quantities. The values of k and ϵ at the inlet of the 1D domain are obtained by linearly extrapolating the DNS values from the location just upstream of the shock to the inlet station, as mentioned in Ref. [21]. The values of k and ϵ at the location just upstream of the shock need to be known *a priori*, and are computed as,

$$k_1 = \frac{M_t^2}{2}, \quad \epsilon_1 = \frac{5 M_t^3}{\sqrt{3} (k_0 \lambda^*) Re_\lambda}, \quad (42)$$

where λ^* is the dimensional Taylor lengthscale. The values of M_t and Re_λ are set as per the problem requirement.

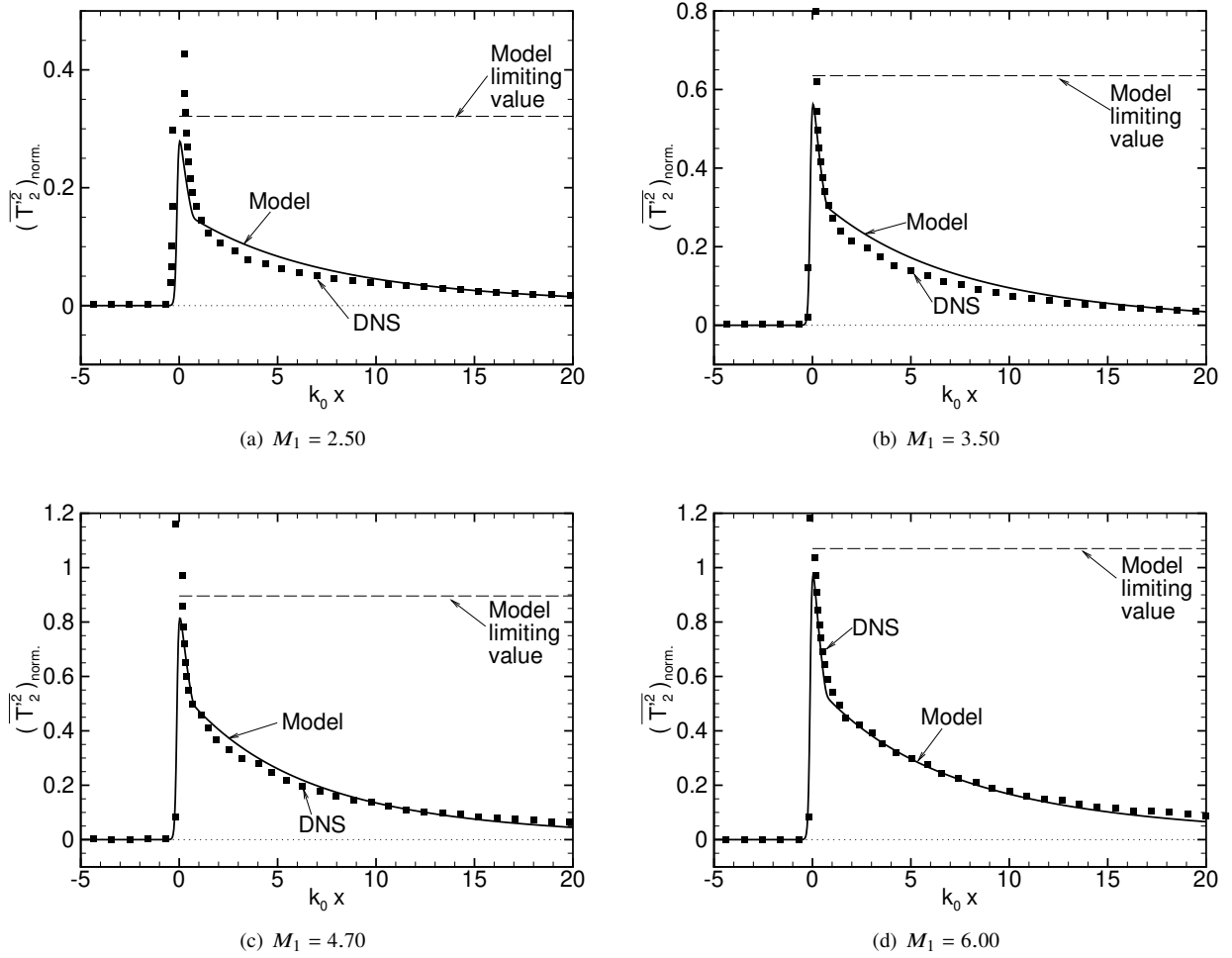


Fig. 9 Variation of normalized temperature variance for the four different Mach numbers. The temperature variance is normalized by $\bar{T}_2^2 (k_1/\bar{u}_1^2)$. Lines correspond to the solution of the RK4 integration of the temperature variance model equation in Eq. 33 and square symbols correspond to the DNS results. The horizontal dashed line is the limiting value for the model which is obtained by integrating Eq. 33 without the decay mechanisms.

A. Temperature variance

Figure 9 compares the model prediction for temperature variance against DNS data for the four different Mach numbers. The switch between the acoustic decay mechanism and the viscous decay mechanism, enables us to capture the spatial variation observed in the DNS profiles. The slope of the rapid decay is controlled by the λ_0 parameter, which is chosen to be $0.75/k_0$. Accurate representation of the rapid decay profile is possible if the lengthscale of transition is known already. The acoustic decay coefficient is varied from 1 for $M_1 = 6$ to 3 for $M_1 = 2.5$ with the viscous decay coefficient kept around a value of 1. The model predicts the post-shock variation of temperature variance accurately.

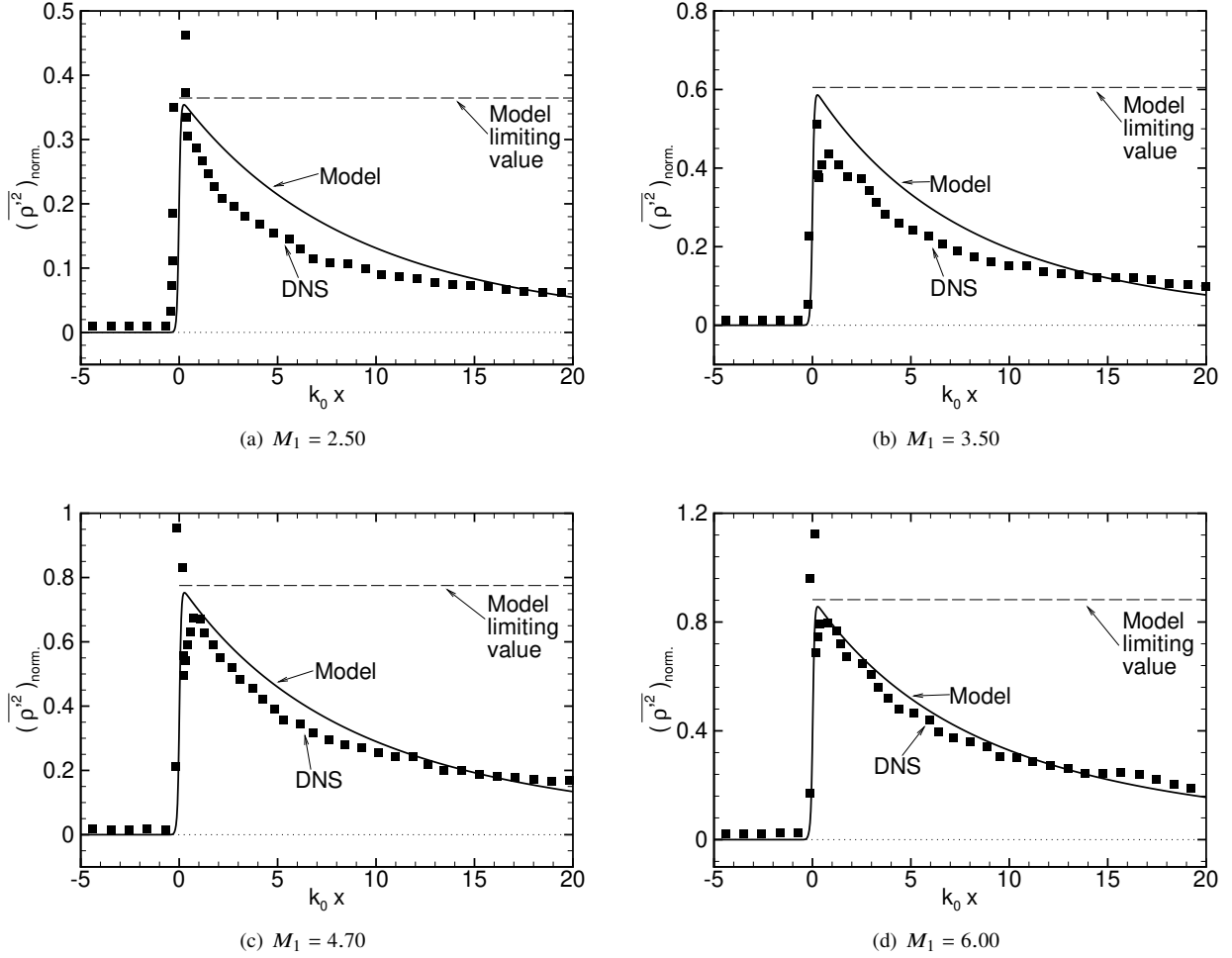


Fig. 10 Variation of normalized density variance for the four different Mach numbers. The density variance is normalized by $\overline{\rho}_2^2 (k_1/\overline{u}_1^2)$. Lines correspond to the solution of the RK4 integration of the model equation in Eq. 33 and square symbols correspond to the DNS results. The horizontal dashed line is the limiting value for the model which is obtained by integrating Eq. 33 without the decay mechanism.

B. Density variance

Figure 10 plots the density variance model against the DNS data for the four different Mach numbers. The viscous decay coefficient is varied from 0.6 to 1.2 with the highest value used for $M_1 = 2.5$ and the lowest value for the high Mach numbers. The excellent match at $M_1 = 6$ indicate that the viscous decay seems to be the sole decay mechanism for the density variance at high Mach numbers. Similar to Reynolds stress profiles, the density variance also show a nonmonotonic variation with a post-shock peak value located at $k_0 x \approx 1$, but only for the high Mach number cases. The amplification behind the shock, predicted with only the production term as the source term matches well with the post-shock peak values for the high Mach number cases considered here. The decay profile at low Mach numbers (not shown) is predicted well with a mixed type of decay profile, as was used for the temperature variance.

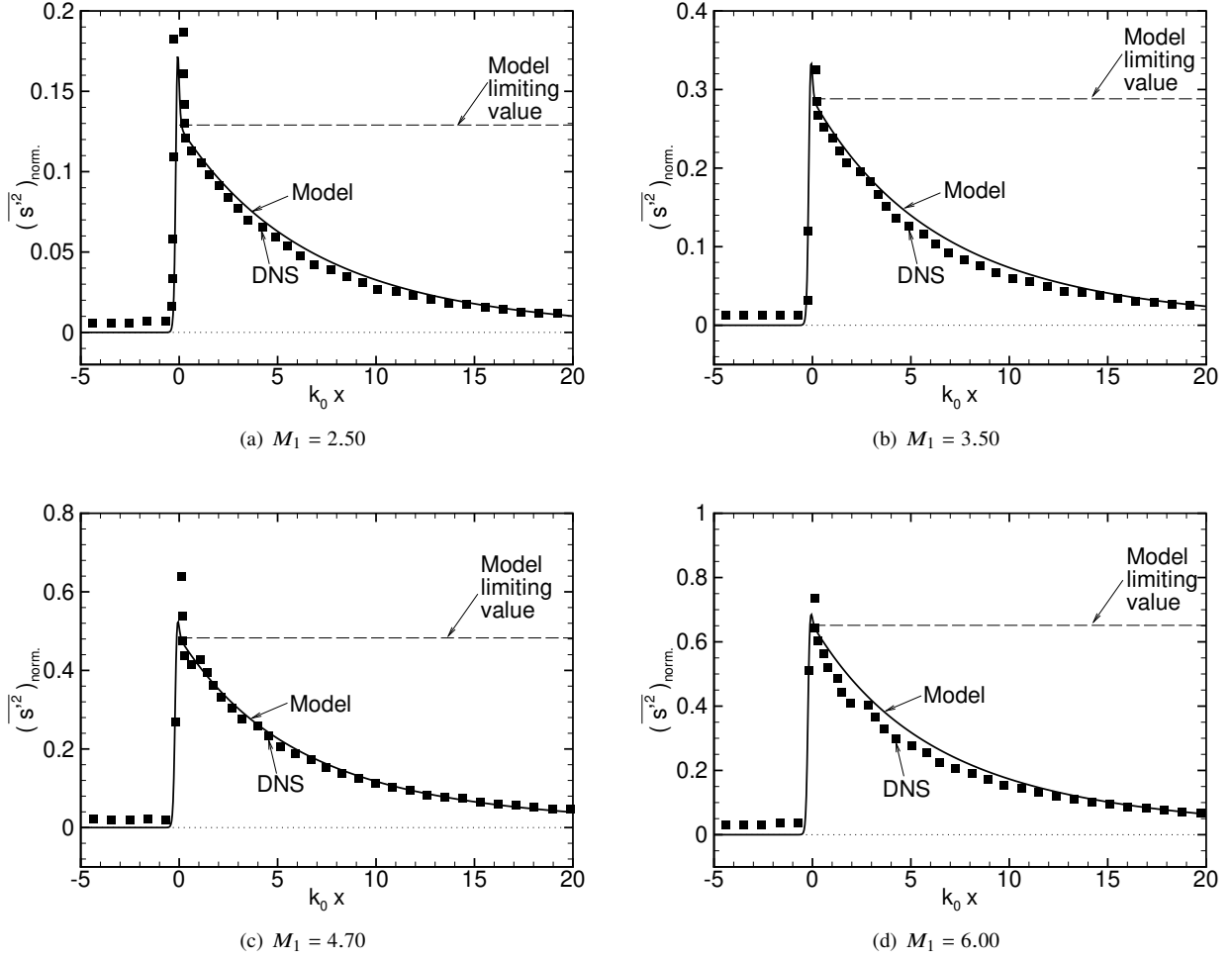


Fig. 11 Variation of normalized entropy variance for the four different Mach numbers. The entropy variance is normalized by $c_p^2 (k_1/\bar{u}_1^2)$. Lines correspond to the solution of the RK4 integration of the model equation in Eq. 33 and square symbols correspond to the DNS results. The horizontal dashed line is the limiting value for the model which is obtained by integrating Eq. 33 without the decay mechanism.

C. Entropy variance

The variation of the modeled entropy variance is compared with its corresponding DNS profile in Fig. 11 for the varying Mach number cases. For the modeling coefficients used in the production term and the decay term (given in Eq. 39), the predictions were found to match well with the DNS data. The modeled production of entropy variance yields a sharp increase in entropy variance at the shock followed by a small decay. This is due to the quadratic dependence between the mean entropy gradient and the mean velocity gradient (see Eq. 21). The model prediction without the decay mechanism is not the sharp peak, but the value denoted by the horizontal dashed line.

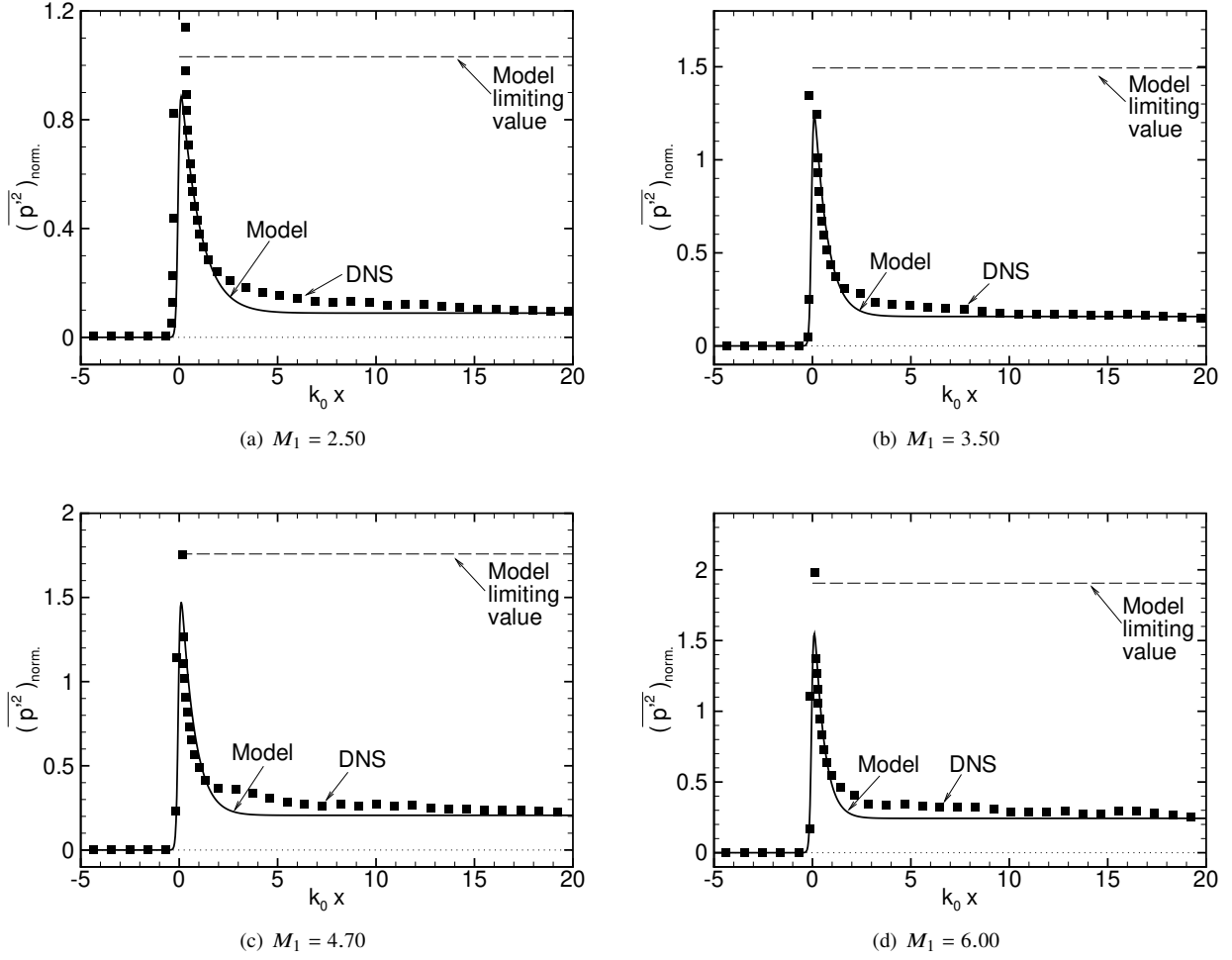


Fig. 12 Variation of normalized pressure variance for the four different Mach numbers. The pressure variance is normalized by $\overline{p_2^2} (k_1/\bar{u}_1^2)$. Lines correspond to the solution of the RK4 integration of the model equation in Eq. 33 and square symbols correspond to the DNS results. The horizontal dashed line is the limiting value for the model which is obtained by integrating Eq. 33 without the decay mechanism.

D. Pressure variance

Unlike the other variances the model for pressure variance require a slightly different type of decay mechanism. Usage of acoustic decay mechanism in the form of $\overline{p^2}/L_\epsilon$ results in the pressure variance tending to zero in the farfield. LIA and DNS both show a finite value for the pressure variance in the farfield region. We make use of LIA to prescribe the farfield value of pressure variance and apply a lower limit on the model. This type of modeling the pressure variance is found to match the DNS data very well as shown in Fig. 12. The model follows the rapid decay of pressure variance behind the shock and asymptote to the LIA prescribed farfield value.

VI. Conclusion

In this paper, we propose predictive models for density, pressure, temperature and entropy variances for the case of canonical shock-turbulence interaction. The models are developed as functions of the turbulent kinetic energy via the turbulent energy flux model of Quadros & Sinha [16]. The accuracy of the models depend on the accuracy of the model used for computing k and ϵ , where the SU $k - \epsilon$ model of Sinha et al. [13] is found to be satisfactory. The underlying physical mechanisms in the evolution of thermodynamic variances behind the shock were highlighted and modeled accordingly. The use of mixed type of decay mechanisms is found to predict variances composed of multiple Kovászny modes. A switching function based on the hyperbolic tangent function is used to employ the mixed type of decay mechanism. The proposed models predict the post-shock amplification and the decay of their respective variances very well for the range of Mach numbers considered in this study. Effects of upstream entropy and acoustic fluctuations, extension of model to three-dimensional framework, and modification for converging/moving spherical shocks (RMI due to shocks) are possible directions for future work.

Appendix

A. Model for temperature flux

The model for the turbulent temperature flux given in Sec. IV, Eq. 8 is derived here. This is similar to the model development procedure used in Ref. [16]. First, we start with the linearized RH relation for the total enthalpy in the frame of reference attached to the instantaneous shock,

$$c_p \frac{\partial T'}{\partial x} = -\bar{u} \frac{\partial u'}{\partial x} - u' \frac{\partial \bar{u}}{\partial x} + \xi_t \frac{\partial \bar{u}}{\partial x}, \quad (43)$$

which upon taking a moment with u' gives the linearized equation for temperature flux as,

$$c_p \frac{\partial}{\partial x} (\overline{u'T'}) = -\bar{u} \frac{\partial}{\partial x} \left(\frac{\overline{u'^2}}{2} \right) - \overline{u'^2} \frac{\partial \bar{u}}{\partial x} + \overline{u'\xi_t} \frac{\partial \bar{u}}{\partial x} + c_p \overline{T' \frac{\partial u'}{\partial x}}, \quad (44)$$

where the first two terms on the RHS are the production terms, the third term is the shock unsteadiness effect and the fourth term is the temperature-dilatation correlation.

Assuming $\overline{u'^2} = 2k/3$ and $\overline{u'\xi_t} = b_1 \overline{u'^2}$ (from [13]), Eq. 44 is rewritten in terms of turbulent kinetic energy, k , where $b_1 = 0.4 + 0.6 e^{2(1-M_1)}$ is the modeling coefficient obtained using LIA. The modified equation reads as,

$$c_p \frac{\partial}{\partial x} (\overline{u'T'}) = -\frac{4}{9} (1 - b_1) k \frac{\partial \bar{u}}{\partial x} + c_p \overline{T' \frac{\partial u'}{\partial x}}, \quad (45)$$

where the model equation for k from Ref. [13],

$$\bar{u} \frac{\partial k}{\partial x} = -\frac{2}{3} [1 - b_1] k \frac{\partial \bar{u}}{\partial x}, \quad (46)$$

is used to solve for the turbulent kinetic energy.

Similar to the procedure used in Ref. [13] to determine the amplification of turbulent kinetic energy, we can compute the amplification of the turbulent temperature flux by considering the production term alone. The modeled equation for $\overline{u'T'}$, thus reduces to,

$$c_p \frac{\partial}{\partial x} (\overline{u'T'}) \approx -\frac{4}{9} (1 - b_1) k \frac{\partial \bar{u}}{\partial x}. \quad (47)$$

Integrating Eq. 46 across the shock yields the relation,

$$\frac{k_2}{k_1} = \left(\frac{\bar{u}_2}{\bar{u}_1} \right)^{-\frac{2}{3} (1-b_1)}. \quad (48)$$

which is a function of the upstream Mach number only, and can be used to obtain a closed form solution for the temperature flux across the shock. Upon substitution of Eq. 48 in Eq. 47 and integrating across the shock yields,

$$c_p (\overline{u_2'T_2'} - \overline{u_1'T_1'}) \approx \left[\frac{4}{9} (1 - b_1) \right] \left(\frac{3}{2b_1 + 1} \right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2} \right)^{-\frac{1}{3} (2b_1+1)} \right] k_1 \bar{u}_1, \quad (49)$$

where $\overline{u_1' T_1'} = 0$ for the case of purely vortical turbulence upstream of the shock. Considering the incoming turbulence to be purely vortical in nature and normalizing Eq. 49 as shown below, the normalized post-shock turbulent temperature flux is obtained as,

$$\left(\overline{u_2' T_2'}\right)_{norm.} = \frac{\overline{u_2' T_2'}}{\bar{u}_1 \bar{T}_2 \left(\frac{k_1}{\bar{u}_1^2}\right)} \approx \underbrace{\left[\frac{4}{9}(1-b_1)\right] \left(\frac{3}{2b_1+1}\right) \left[1 - \left(\frac{\bar{u}_1}{\bar{u}_2}\right)^{-\frac{1}{3}(2b_1+1)}\right]}_{C_{uT}} (\gamma-1) \frac{M_1^2}{(\bar{T}_2/\bar{T}_1)}, \quad (50)$$

where the normalized downstream temperature flux is a function of the Mach number and the ratio of specific heats only. The coefficient C_{uT} can thus be used to model the temperature flux as,

$$\overline{u' T'} = C_{uT} \frac{k \bar{u}}{c_p}, \quad (51)$$

where Eq. 49 is used as the guiding principle.

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References

- [1] Lele, S. K., “Compressibility Effects on Turbulence,” *Annu. Rev. Fluid Mech.*, Vol. 26, No. 1, 1994, pp. 211–254.
- [2] Bhagatwala, A. V., “Shock-turbulence interaction and Richtmyer-Meshkov instability in spherical geometry,” Ph.D. thesis, Stanford University, 2011.
- [3] Soullard, O., Griffond, J., and Souffland, D., “Pseudocompressible approximation and statistical turbulence modeling: Application to shock tube flows,” *Phys. Rev. E*, Vol. 85, 2012, p. 026307.
- [4] Abdikamalov, E., Zhaksylykov, A., Radice, D., and Berdibek, S., “Shock–turbulence interaction in core-collapse supernovae,” *Mon. Not. R. Astron. Soc.*, Vol. 461, No. 4, 2016, pp. 3864–3876.
- [5] Huete, C., Abdikamalov, E., and Radice, D., “The impact of vorticity waves on the shock dynamics in core-collapse supernovae,” *Mon. Not. R. Astron. Soc.*, Vol. 475, No. 3, 2018, pp. 3305–3323.
- [6] Ribner, H. S., “Acoustic energy flux from shock-turbulence interaction,” *J. Fluid Mech.*, Vol. 35, No. 2, 1969, pp. 299–310.
- [7] Ribner, H. S., “Spectra of noise and amplified turbulence emanating from shock-turbulence interaction,” *AIAA J.*, Vol. 25, No. 3, 1987, pp. 436–442.
- [8] Quadros, R., Sinha, K., and Larsson, J., “Turbulent energy flux generated by shock/homogeneous-turbulence interaction,” *J. Fluid Mech.*, Vol. 796, 2016, pp. 113–157.
- [9] Mahesh, K., Moin, P., and Lele, S. K., “The interaction of a shock wave with a turbulent shear flow,” Tech. Rep. TF-69, Thermosciences Division, Department of Mechanical Engineering, Stanford University, Stanford, California 94305, June 1996.
- [10] Larsson, J., Bermejo-Moreno, I., and Lele, S. K., “Reynolds-and Mach-number effects in canonical shock-turbulence interaction,” *J. Fluid Mech.*, Vol. 717, 2013, pp. 293–321.
- [11] Larsson, J., and Lele, S. K., “Direct numerical simulation of canonical shock/turbulence interaction,” *Phys. Fluids*, Vol. 21, No. 12, 2009, p. 126101.
- [12] Sethuraman, Y. P. M., Sinha, K., and Larsson, J., “Thermodynamic fluctuations in canonical shock-turbulence interaction: effect of shock strength,” *Theor. Comput. Fluid Dyn.*, Vol. 32, No. 5, 2018, pp. 629–654.
- [13] Sinha, K., Mahesh, K., and Candler, G. V., “Modeling shock unsteadiness in shock/turbulence interaction,” *Phys. Fluids*, Vol. 15, No. 8, 2003, pp. 2290–2297.
- [14] Veera, V. K., and Sinha, K., “Modeling the effect of upstream temperature fluctuations on shock/homogeneous turbulence interaction,” *Phys. Fluids*, Vol. 21, No. 2, 2009, p. 025101.

- [15] Mahesh, K., Lele, S. K., and Moin, P., “The influence of entropy fluctuations on the interaction of turbulence with a shock wave,” *J. Fluid Mech.*, Vol. 334, 1997, pp. 353–379.
- [16] Quadros, R., and Sinha, K., “Modelling of turbulent energy flux in canonical shock-turbulence interaction,” *Int. J. Heat Fluid Flow*, Vol. 61, Part B, 2016, pp. 626–635.
- [17] Quadros, R., Sinha, K., and Larsson, J., “Kovasznyai Mode Decomposition of Velocity-Temperature Correlation in Canonical Shock-Turbulence Interaction,” *Flow, Turbul. and Combust.*, Vol. 97, No. 3, 2016, pp. 787–810.
- [18] Kovásznyai, L. S. G., “Turbulence in Supersonic Flow,” *J. Aeronaut. Sci.*, Vol. 20, No. 10, 1953, pp. 657–674, 682. Reprinted in *AIAA J.*, **41**(7), 219–237, 2003.
- [19] Honein, A. E., and Moin, P., “Higher entropy conservation and numerical stability of compressible turbulence simulations,” *J. Comput. Phys.*, Vol. 201, No. 2, 2004, pp. 531–545.
- [20] Sinha, K., “Evolution of enstrophy in shock/homogeneous turbulence interaction,” *J. Fluid Mech.*, Vol. 707, 2012, pp. 74–110.
- [21] Raje, P., and Sinha, K., “A physically consistent and numerically robust $k - \epsilon$ model for computing turbulent flows with shock waves,” *Comput. Fluids*, Vol. 136, 2016, pp. 35–47.