Linear stability of boundary layers in hypersonic flows

Ashwin Ramachandran

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Indian Institute of Technology, Bombay

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Overview

1 Boundary Layer in Hypersonic flows

2 Modal analysis of compressible plane Couette flow

Boundary layers in hypersonic flows

Features of boundary layers

- The temperature is high enough for chemical reactions to take place
- High temperature gradients with varying viscosity
- Thermal vibration modes excited
- BL edge conditions depend on geometry - bluff/slender body

Hypersonic flows - Boundary layer

Governing equations

Conservation of total mass:

$$\frac{\partial}{\partial s}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

Conservation of species:

$$\rho u \frac{\partial C_i}{\partial s} + \rho v \frac{\partial C_i}{\partial y} = \frac{\partial}{\partial y}\left(\rho D \frac{\partial C_i}{\partial y}\right) + w_i \tag{2}$$

The diffusive flux due to concentration gradients is given by Fick's law given by

$$V_{iy} = -\frac{D}{C_i}\frac{\partial C_i}{\partial y}$$

Hypersonic flows - Boundary layer

Governing equations

Momentum equation - s direction:

$$\rho u \frac{\partial u}{\partial s} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial s} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{3}$$

Momentum equation - y direction:

$$\frac{\partial p}{\partial y} = 0 \;\; or \;\; p = p_e(s) \tag{4}$$

Hypersonic flows - Boundary layer

Governing equations

Conservation of vibrational energy:

$$\frac{\partial (E_v u)}{\partial s} + \frac{\partial (E_v v)}{\partial y} = \frac{\partial}{\partial y}\left(k_v \frac{\partial T_v}{\partial y}\right) - \sum_i^{nd} \frac{\partial}{\partial y}(E_{vi} V_{ij}) + w_v \quad (5)$$

The total vibrational energy E_v is related to the species vibrational energy E_{vi} by

$$E_v = \sum_i E_{vi} = \sum_i \rho_i \frac{R}{M_i}\left(\frac{\theta_{vi}}{e^{\frac{\theta_{vi}}{T_v}} - 1}\right) = \rho R \sum_i^{nd} \frac{C_i}{M_i}\left(\frac{\theta_{vi}}{e^{\frac{\theta_{vi}}{T_v}} - 1}\right) \quad (6)$$

Hypersonic flows - Boundary layer

Governing equations

Conservation of total energy:

$$\rho u \frac{\partial H}{\partial s} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} + k_v \frac{\partial T_v}{\partial y} + \rho D \sum_i h_i \frac{\partial C_i}{\partial y} + \frac{\mu}{2} \frac{\partial u^2}{\partial y} \right) \tag{7}$$

where

$$H = h + \frac{u^2}{2} \tag{8}$$

$$h = \sum_i C_i h_i \tag{9}$$

$$h_i = C_{p_i} dT + C_{p_{v,i}} dT_v + h_i^0 \tag{10}$$

Similarity solution to boundary layers

Similarity approach

- Convert the PDEs to ODEs by suitable transformation of variables
- Assume a self similar boundary layer profile
- Using Levy-Lees co-ordinate transformation as given in Dorrance (1957)
- Solution to the similarity variables under the assumption that the flow satisfies similarity conditions

Similarity solution to boundary layers

Introduce the stream function defined by

$$\psi(s,y) = \int_0^y \frac{u}{\bar{R}T} dy + \psi(s,0) \tag{11}$$

Assume a similar solution

$$\psi(s,\eta) = N(s)\, f(\eta) \tag{12}$$

and

$$u(s,\eta) = u_e(s)\, f'(\eta) \tag{13}$$

$$C_i = (C_i)_e\, z_i(\eta) \tag{14}$$

$$T_v = T_{ve}\, g(\eta) \tag{15}$$

$$T = T_e I(\eta) \tag{16}$$

Similarity solution

Solutions - Momentum, Species conservation

$$\left(Cf''\right)' + ff'' + \frac{2\bar{s}}{u_e}\frac{du_e}{ds}\left[\frac{\rho_e}{\rho} - (f')^2\right] = 0 \tag{17}$$

$$\left(\frac{C}{S}z_i'\right)' + fz_i' = \frac{2\bar{s}f'z_i}{(C_i)_e}\frac{d\left(C_i\right)_e}{d\bar{s}} - \frac{2\bar{s}\dot{w}_i}{\rho u_e^2\rho_e\mu_e\left(C_i\right)_e} \tag{18}$$

where, $C = \frac{\rho\mu}{\rho_e\mu_e}$ is the Chapman-Rubesin coefficient.The above equation is a non-linear ODE in f, in one variable η.

Similarity solution

Solution - Vibrational Energy

$$\left(\frac{C}{Pr_v}\frac{C_{p,ref}}{R}g'\right)' + \frac{f}{T_{ve}}\sum_i^{nd}\left\{C_{ie}\frac{\theta_{vi}}{M_i}\left(\frac{z_i}{e^{\left(\frac{\theta_{vi}}{T_{ve}}\right)\frac{1}{g}}-1}\right)'\right\}$$
$$= \frac{2\bar{s}f'}{T_{ve}}\sum_i^{nd}\left\{\frac{\theta_{vi}}{M_i}\left(\frac{z_i}{e^{\left(\frac{\theta_{vi}}{T_{ve}}\right)\frac{1}{g}}-1}\right)\frac{dC_{ie}}{d\bar{s}}\right\} - \frac{1}{T_{ve}}\sum_i^{nd}\left\{\frac{C}{S}\frac{\theta_{vi}}{M_i}\right. \quad (19)$$
$$\left.\left(\frac{C_{ie}z_i'}{e^{\left(\frac{\theta_{vi}}{T_{ve}}\right)\frac{1}{g}}-1}\right)\right\}' - \frac{2\bar{s}\dot{w}_v}{R\rho u_e^2\rho_e\mu_e T_{ve}}$$

Similarity solution

Solution - Total Energy

$$\begin{aligned}
&\left(\frac{C}{Pr}\frac{C_{p,ref}}{R}l'\right)' + f\left\{\frac{u_e^2}{RT_e}f'f'' + \left(l\sum_i \frac{C_{p,i}}{R}C_{ie}z_i\right)' + \frac{T_{ve}}{T_e}\left(g\sum_i \frac{C_{p_{v,i}}}{R}C_{ie}z_i\right)' + \right. \\
&\left.\sum_i z_i' C_{ie}\frac{h_i^0}{RT_e}\right\} = \frac{2\bar{s}}{T_e}f'\left\{\frac{\left(f'\right)^2}{2R}\frac{du_e^2}{d\bar{s}} + \left(l\frac{dT_e}{d\bar{s}}\sum_i \frac{C_{p,i}}{R}C_{ie}z_i\right)\right. \\
&\left.+\left(g\frac{dT_{ve}}{d\bar{s}}\sum_i \frac{C_{p_{v,i}}}{R}C_{ie}z_i\right)' + \sum_i z_i\frac{h_i^0}{R}\frac{dC_{ie}}{d\bar{s}}\right\} - \frac{T_{ve}}{T_e}\left(\frac{C}{Pr_v}\frac{C_{p,ref}}{R}g'\right)' - \left\{\frac{C}{S}\right. \\
&\left.\sum_i C_{ie}z_i'\left(\frac{C_{p,i}}{R}l + \frac{C_{p_{v,i}}}{R}\frac{T_{ve}}{T_e}g + \frac{h_i^0}{RT_e}\right)\right\}' - \frac{u_e^2}{RT_e}\left(Cf'f''\right)'
\end{aligned} \tag{20}$$

Source terms

- Chemical source terms
- Vibrational source terms

Similarity forms

- Use a "reaction model" along with models for reaction rates
- Use a model for vibrational energy sources in terms of the molecular dynamics
- Express all the quantities in terms of the similarity forms and use them inthe equations
- Results depend on the transport models used

Similarity conditions

- $\frac{2\bar{s}}{u_e}\frac{du_e}{d\bar{s}} = constant$, and $\frac{\rho_e}{\rho} = fn\left(\eta\right)$ only.
- $\frac{2\bar{s}}{C_{ie}}\frac{dC_{ie}}{d\bar{s}} = constant$ and $\frac{2\bar{s}\dot{w}_i}{\rho u_e^2 \rho_e \mu_e (C_i)_e} = constant$.
- $\frac{2\bar{s}}{T_{ve}}\frac{dC_{ie}}{d\bar{s}} = constant$, $T_{ve} = constant$, $\frac{C_{ie}}{T_{ve}} = constant$ and $\frac{2\bar{s}\dot{w}_v}{R\rho u_e^2 \rho_e \mu_e T_{ve}} = constant$.
- $2\bar{s}\frac{u_e^2}{RT_e} = constant$, $C_{ie} = constant$, $\frac{T_{ve}}{T_e} = constant$, $\frac{2\bar{s}}{T_e}\frac{du_e^2}{d\bar{s}} = constant$, $\frac{2\bar{s}}{T_e}\frac{dT_e}{d\bar{s}} = constant$, $\frac{2\bar{s}}{T_e}\frac{dT_{ve}}{d\bar{s}} = constant$, $\frac{2\bar{s}}{T_e}\frac{dC_{ie}}{d\bar{s}} = constant$ and $\frac{C_{ie}}{T_e} = constant$.

Simplifications for a flat plate boundary layer

Momentum equation

$$\left(Cf''\right)' + ff'' = 0 \tag{21}$$

Species conservation

$$\left(\frac{C}{S}z_i'\right)' + fz_i' = -\frac{2\bar{s}\dot{w}_i}{\rho u_e^2 \rho_e \mu_e \left(C_i\right)_e}$$

and if we further simplify the above for a frozen flow ($\dot{w}_i = 0$), we have

$$\left(\frac{C}{S}z_i'\right)' + fz_i' = 0 \tag{22}$$

Simplifications for a flat plate boundary layer

Vibrational energy equation

$$\left(\frac{C}{Pr_v}\frac{C_{p,ref}}{R}g'\right)' + \frac{f}{T_{ve}}\sum_i^{nd} C_{ie}\frac{\theta_{vi}}{M_i}\left(\frac{z_i}{e^{\left(\frac{\theta_{vi}}{T_{ve}}\right)\frac{1}{g}} - 1}\right)'$$
$$= -\frac{1}{T_{ve}}\sum_i^{nd}\left\{\frac{C}{S}\frac{\theta_{vi}}{M_i}\left(\frac{C_{ie}z_i'}{e^{\left(\frac{\theta_{vi}}{T_{ve}}\right)\frac{1}{g}} - 1}\right)\right\}' - \frac{2\bar{s}\dot{w}_v}{R\rho u_e^2\rho_e\mu_e T_{ve}} \tag{23}$$

$\dot{w}_v = Q_{T-V} + Q_{V-D}$. For no vibrational energy source, we have

$$\left(\frac{C}{Pr_v}\frac{C_{p,ref}}{R}g'\right)' + \sum_i^{nd}\frac{C_{ie}}{M_i}\left(\frac{\theta_{vi}}{T_{ve}}\right)^2\frac{e^{\frac{\theta_{vi}}{g\,T_{ve}}}}{\left(e^{\frac{\theta_{vi}}{g\,T_{ve}}} - 1\right)^2}\left\{\frac{fz_ig' + \frac{C}{S}z_i'g'}{g^2}\right\} = 0 \tag{24}$$

Simplifications for a flat plate boundary layer

Total energy equation

$$\left(\frac{C}{Pr}\frac{C_{p,ref}}{R}l'\right)' + f\left\{\frac{u_e^2}{RT_e}f'f'' + \left(l\sum_i \frac{C_{p,i}}{R}C_{ie}z_i\right)' + \frac{T_{ve}}{T_e}\left(g\sum_i \frac{C_{p_{v,i}}}{R}C_{ie}z_i\right)' + \right\}$$
$$= -\frac{T_{ve}}{T_e}\left(\frac{C}{Pr_v}\frac{C_{p,ref}}{R}g'\right)' - \left\{\frac{C}{S}\sum_i C_{ie}z_i'\left(\frac{C_{p,i}}{R}l + \frac{C_{p_{v,i}}}{R}\frac{T_{ve}}{T_e}g + \frac{h_i^0}{RT_e}\right)\right\}' - \frac{u_e^2}{RT_e}\left(Cf'f''\right)' \tag{25}$$

For the assumption of a frozen flow, the above equation reduces to

$$\left(\frac{C}{Pr}\frac{C_{p,ref}}{R}l'\right)' + \left(\frac{C}{Pr_v}\frac{C_{p,ref}}{R}\frac{T_{ve}}{T_e}g'\right)' + \frac{u_e^2}{RT_e}Cf''^2 + \sum_i C_{ie}\left(\frac{C_{p,i}}{R}l' + \frac{C_{p,vi}}{R}\frac{T_{ve}}{T_e}g'\right)\left(fz_i + \frac{C}{S}z_i'\right) = 0 \tag{26}$$

Hypersonic boundary layer stability

Need for stability

- The transition from laminar to turbulent flow significantly affects the aerodynamic performance and surface heating of a hypersonic vehicle
- The skin friction and surface heating are higher for a turbulent boundary layer than a laminar boundary layer
- Accurate prediction of transition in the boundary layer helps in optimum design of the space vehicle and the TPS

Application

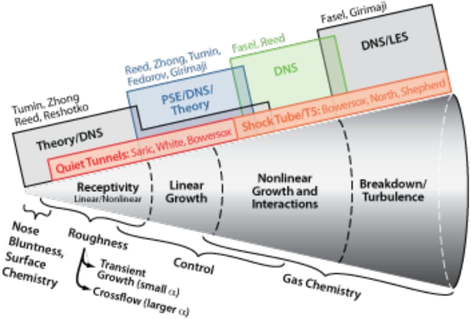

Figure : Hypersonic transition schematic (taken from TAMU website)

Hypersonic boundary layer stability

Methodology

- Linearise the full form of the base flow
- Use normal mode forms
  $[u', v', w', p', T', T'_v, C'_i] =$
  $[f_1(y), f_2(y), f_3(y), f_4(y), f_5(y), f_6(y), g_i(y)]\, e^{\iota(\alpha s+\beta z-\omega t)}$
- Set up an eigenvalue problem

Hypersonic boundary layer stability

Setting up the eigenvalue problem

For eg., continutiy equation in normal mode form is

$$\iota\left(\bar{u}\alpha-\omega\right)\left(\frac{f_4}{\bar{p}}-\frac{f_5}{\bar{T}}\right)+\iota\alpha f_1+\iota\beta f_3+\frac{f_2}{\bar{\rho}}\frac{d\bar{\rho}}{dy}+f_2'=0 \quad (27)$$

Similarly obtain for other equations.
Set up the genralised eigenvalue system defined as

$$B\bar{x}=\iota\,\omega A\bar{x} \quad (28)$$

where

$$\bar{x}=\left\{f_1\;\; f_2\;\; f_4\;\; f_4\;\; f_5\;\; f_6\;\; g_i\right\}^T$$

is the pertubation variables vector.

Modal analysis of compressible plane Couette flow

General Base flow

- Continuity and momentum in z are identically satisfied.
- Momentum equation in y direction gives pressure is constant in the y direction
- Conservation of momentum in x direction reduces to

$$\frac{d}{dy}\left[\mu\frac{du}{dy}\right]=0 \tag{29}$$

- Energy conservation gives

$$\frac{1}{Pr}\left[\frac{d}{dy}\left(k\frac{dT}{dy}\right)\right]+(\gamma-1)\,M^2\mu\left(\frac{\partial u}{\partial y}\right)^2=0 \tag{30}$$

Pr is a constant based on the reference values

Isolating the effect of viscosity stratification and thermal conductivity on stability

Different cases

- Case 1: Constant thermal conductivity, viscosity constant
- Case 2: Varying thermal conductivity, viscosity constant
- Case 3: Varying thermal conductivity, varying viscosity
  - 3a: Thermal conductivity and viscosity variations are dependent on each other
  - 3b: Thermal conductivity and viscosity vary independently
- Case 4: Constant thermal conductivity, varying viscosity

Case 1: Constant thermal conductivity, viscosity constant

Base flow

Flow field is given by

$$\left[\frac{d}{dy}\left(\frac{dT}{dy}\right)\right] + (\gamma - 1)\,Pr\,M^2 = 0 \tag{31}$$

which on double integration with respect to y gives

$$T = -\frac{1}{2}(\gamma - 1)\,Pr\,M^2 y^2 + c_1 y + c_2 \tag{32}$$

where c_1 and c_2 can be determined by the boundary conditions. Constant viscosity assumption gives $\mu = 1$ and

$$u = y. \tag{33}$$

Case 2: Varying thermal conductivity, viscosity constant

Base flow

Require another equation relating k and T in order to solve the system. Flow field given by

$$u = y \tag{34}$$

$$\frac{d}{dy}\left(k\frac{dT}{dy}\right) + (\gamma - 1)\, Pr\, M^2 = 0 \tag{35}$$

$$\bar{k} = A\left(\frac{T^{3/2}}{B + T}\right) \tag{36}$$

where A and B are constants based on a conductivity-temperature correlation model (similar to Sutherland's law). Equations 35 and 36 can be solved together numerically.

Case 3a: Thermal conductivity and viscosity variations are dependent on each other

Base flow

k and μ are related by the Pr.
For a constant Pr,we can choose either of μ or k to vary independently according to a constitutive model.

- The case of μ variation independently according to Sutherlands's law

$$\bar{\mu} = A_2 \left(\frac{T^{3/2}}{B_2 + T} \right)$$

and then k obtained as

$$k = \frac{C_p}{Pr} \mu$$

has been done by Malik et al. (2006)

Case 3a: Thermal conductivity and viscosity variations are dependent on each other

Base flow

- We could also let the dependence of $\mu$and k be based on k varying independently as

$$\bar{k} = A_1 \left(\frac{T^{3/2}}{B_1 + T} \right)$$

and then varying μ depending on k by

$$\mu = \frac{Pr}{C_p} k.$$

In this case, the governing equations can be re-written in terms of only μ or k.

Case 3b: Thermal conductivity and viscosity vary independently

Base flow

k and μ are follow their constitutive relation with temperature.

$$\frac{d}{dy}\left[\mu\frac{du}{dy}\right] = 0$$

$$\frac{1}{Pr}\left[\frac{d}{dy}\left(k\frac{dT}{dy}\right)\right] + (\gamma - 1)\, M^2\mu\left(\frac{\partial u}{\partial y}\right)^2 = 0$$

$$\bar{\mu} = A_2\left(\frac{T^{3/2}}{B_2 + T}\right)$$

$$\bar{k} = A_1\left(\frac{T^{3/2}}{B_1 + T}\right)$$

which can be solved numerically.

Case 4: Constant thermal conductivity, varying viscosity

Base flow

Flow field is given by

$$\frac{du}{dy} = \frac{\tau}{\mu(y)} \tag{37}$$

where $\tau = constant$ is the shear stress and is determined by satisfying the boundary conditions $u(0) = 0$ and $u(1) = 1$. The energy equation reduces to

$$\frac{d}{dy}\left(\frac{dT}{dy}\right) + (\gamma - 1)\ Pr\ M^2 \frac{\tau^2}{\mu(y)} = 0 \tag{38}$$

and viscosity varies according to Sutherland's law, which is

$$\bar{\mu} = A_2 \left(\frac{T^{3/2}}{B_2 + T}\right). \tag{39}$$

Summary of different cases

CASES	$\mu_o,\ k_o$	$\mu_o,\ k(T)$	$\mu(T),\ k(T)$	$\mu(T),\ k_o$
$\mu_o,\ k_o$	$\times$	k effect	*Malik* et al.	μ effect
$\mu_o,\ k(T)$	k effect	$\times$	-	*Malik* et al.
$\mu(T),\ k(T)$	*Malik* et al.	Pr effect	$\times$	Pr effect
$\mu(T),\ k_o$	μ effect	-	Pr effect	$\times$

Table : Summary of different cases

Base flow results for Case 3a - Validation @ M=2,5

Boundary conditions: Lower adiabatic wall, upper isothermal wall

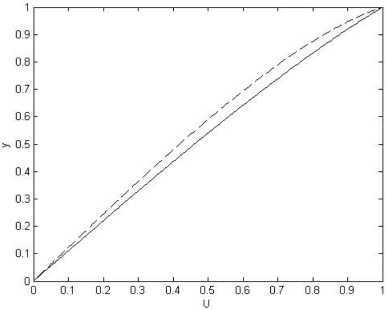

Figure : Velocity

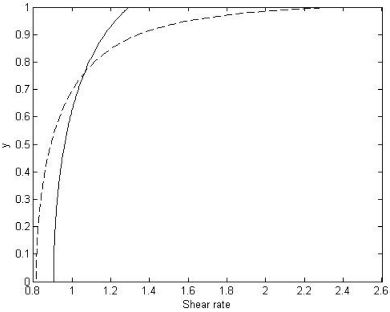

Figure : Shear rate

Base flow results for Case 3a - Validation @ M=2,5

Boundary conditions: Lower adiabatic wall, upper isothermal wall

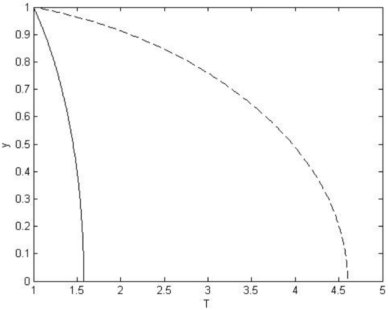

Figure : Temperature

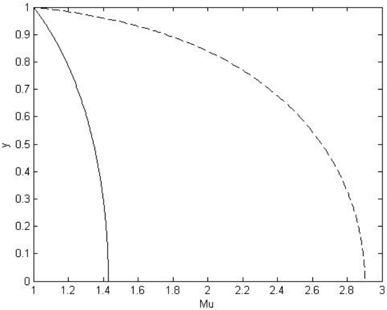

Figure : Viscosity

Preliminary temporal stability results

Comparison with Hu et al.(1998)

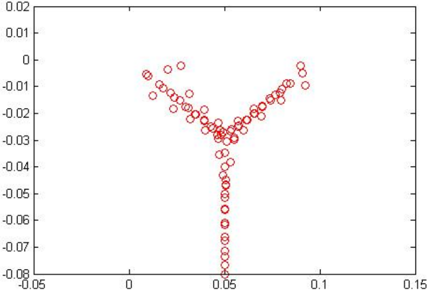

Figure : Phase velocity spectrum at $M = 2$, $Re = 2 \times 10^5$, $\alpha = 0.1$ with 100 grid points

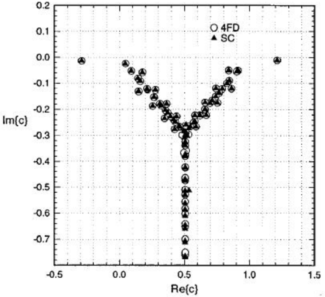

Figure :

References

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Thank you

Extra

Mathematical formulation of the stability problem

General procedure

- Develop and solve for the base flow equations
- Subject it to small perturbations from the steady state
- Assume the perturbations are small enough for linearity
- Assign normal mode form (Fourier waveforms) to the perturbations
- Set up an eigenvalue problem depending on spatial/temporal/spatio-temporal analysis
- Apply appropriate boundary conditions on all the perturbations
- Solve for the eigenvalues to get the amplification rates of the perturbations

Chebyshev spectral collocation method

Spectral methods

We would like to address two issues:

- Given a 1-dimensional domain in space, in what manner do we discretize it?
- Having discretized the domain into points $\{y\}_i$ and knowing the values of a function $\{f(y)\}_i$ at those points, how do we compute the derivatives?

Spectral methods have higher level of accuracy and faster convergence rates compared to the well known differencing methods.

Chebyshev spectral collocation method

Nodes - Chebyshev-Lobotto points or Gauss-Chebyshev-Lobotto points

$$x_k = \cos\left(\frac{\pi}{2}\frac{2k-1}{n}\right), \quad k = 1, \ldots, n. \tag{40}$$

which are the zeros of the chebyshev polynomials

$$T_n(x) = \cos\left(n\cos^{-1}(x)\right). \tag{41}$$

These polynomials are solutions to the Chebyshev differential equation

$$\left(1-x^2\right)y'' - xy' + n^2 y = 0 \tag{42}$$

for $x \in [-1, 1]$.

Derivatives

Chebyshev differentiation matrices

For a function $u(x)$, the n^{th}- derivative at the collocation point x_i is

$$\left(\frac{d^n}{dx^n}u(x)\right)_{x=x_i} = [D^n u(x)]_{i-th\ row}$$

where D is an $n \times n$ matrix whose entries are given by

$$D_{ij} = \begin{cases} \frac{2n^2+1}{6} & : \ i = j = 0 \\ -\frac{2n^2+1}{6} & : \ i = j = n \\ -\frac{x_i}{1-x_i^2} & : \ i = 1, \ldots, n-1 \\ -\frac{c_i}{c_j}\frac{1^{i+j}}{x_i - x_j} & : \ i \neq j, \ i, j = 0, \ldots, n \end{cases}$$

$$c_i = \begin{cases} 2 & : \ i = 0, n \\ 1 & : \ otherwise \end{cases}.$$

Chebyshev differentiation matrices

$$D = \begin{array}{|c|c|c|} \hline \frac{2N^2+1}{6} & 2\frac{(-1)^j}{1-x_j} & \frac{1}{2}(-1)^N \\ \hline -\frac{1}{2}\frac{(-1)^i}{1-x_i} & \begin{array}{ccc} & & \frac{(-1)^{i+j}}{x_i-x_j} \\ & \frac{-x_j}{2(1-x_j^2)} & \\ \frac{(-1)^{i+j}}{x_i-x_j} & & \end{array} & \frac{1}{2}\frac{(-1)^{N+i}}{1+x_i} \\ \hline -\frac{1}{2}(-1)^N & -2\frac{(-1)^{N+j}}{1+x_j} & -\frac{2N^2+1}{6} \\ \hline \end{array}$$

Figure : Chebyshev differentiation matrix for N discretization points.

Scaling of domains - Chebyshev mapping

Finite domain

Suppose that the flow domain is from $y \in [a, b]$, we define a linear map from y to x such that

$$a \mapsto -1,$$

$$b \mapsto 1$$

by the transformation

$$x = -1 + 2\left(\frac{y-a}{b-a}\right) \tag{43}$$

x gives the equivalent Chebyshev nodes and the n^{th}-derivative will then be scaled by a factor of $\left(\frac{2}{b-a}\right)^n$.

Scaling of domains - Chebyshev mapping

Infinite domain

- **Exponential mapping** : The semi-infinite flow domain $0 \leq y < \infty$ is mapped to the computational domain $\eta \in [-1, 1]$ by $\eta = e^{-y/Y}$
- **Algebraic mapping** : In general, the algebraic mappings are of the form $\eta = \frac{a+by}{c+dy}$ with the parameters a, b, c and d accounting for the effects of grid stretching.

Stability equations

We assume the disturbances to be of the form

$$\left[u', v', w', \rho', T'\right] = \left[f_1\left(y\right), f_2\left(y\right), f_3\left(y\right), f_4\left(y\right), f_5\left(y\right)\right] e^{\iota(\alpha x + \beta z - \omega t)} \tag{44}$$

$$\iota\left(\bar{u}\alpha-\omega\right)\left(\frac{f_4}{\bar{\rho}}\right)+\iota\alpha f_1+\iota\beta f_3+\frac{f_2}{\bar{\rho}}\frac{d\bar{\rho}}{dy}+f_2'=0 \tag{45}$$

$$\begin{aligned}\bar{\rho}\left(\iota f_1(\alpha\bar{u}-\omega)+f_2\frac{d\bar{u}}{dy}\right)=&-\frac{\iota\alpha}{\gamma M^2}\left(\bar{T}f_4+\bar{\rho}f_5\right)+\frac{1}{Re}\Big[\big(\bar{\lambda}\\&+2\bar{\mu}\big)\left(-\alpha^2 f_1\right)+\bar{\lambda}\left(\iota\alpha f_2'-f_3\alpha\beta\right)\Big]+\frac{1}{Re}\Bigg[\frac{d\bar{\mu}}{d\bar{T}}\\&\left(f_5\frac{d^2\bar{u}}{dy^2}+\frac{d\bar{u}}{dy}f_5'\right)+\frac{d\bar{u}}{dy}f_5\frac{d\bar{T}}{dy}\frac{d^2\bar{\mu}}{dT^2}+\frac{d\bar{\mu}}{dy}\\&\left(f_1'-\iota\alpha f_2\right)+\bar{\mu}\left(f_1''+\iota\alpha f_2'-\beta^2 f_1-\alpha\beta f_3\right)\Bigg]\end{aligned} \tag{46}$$

$$\bar{\rho}\iota\left(-f_2\omega+\bar{u}f_2\alpha\right)=-\frac{1}{\gamma M^2}\left(\bar{T}f_4'+\bar{\rho}f_5'+f_4\frac{d\bar{T}}{dy}+f_5\frac{d\bar{\rho}}{dy}\right)+\frac{1}{Re}\left[\left(\bar{\lambda}\right.\right.$$
$$\left.\left.+2\bar{\mu}\right)f_2''+\bar{\lambda}\iota\left(f_1'\alpha+f_3'\beta\right)\right]+\frac{1}{Re}\left[\frac{d\left(\bar{\lambda}+2\bar{\mu}\right)}{d\bar{T}}f_2'\right.$$
$$\left.+\frac{d\bar{\lambda}}{d\bar{T}}\iota\left(f_1\alpha+f_3\beta\right)\right]\frac{d\bar{T}}{dy}+\frac{1}{Re}\left[\bar{\mu}\left(\iota\left(\alpha f_1'\right.\right.\right.$$
$$\left.\left.\left.+f_3'\beta\right)-f_2\left(\alpha^2+\beta^2\right)\right)+\iota\frac{d\bar{\mu}}{d\bar{T}}\frac{d\bar{u}}{dy}f_5\alpha\right] \tag{47}$$

$$\bar{\rho}\iota\left(-f_3\omega+\bar{u}f_3\alpha\right)=-\iota\beta\bar{p}\left(\frac{f_4}{\bar{\rho}}+\frac{f_5}{\bar{T}}\right)+\frac{1}{Re}\left[\left(\bar{\lambda}+2\bar{\mu}\right)\left(-\beta^2 f_3\right)+\bar{\lambda}\right.$$
$$\left.\left(\iota\beta f_2'-f_1\alpha\beta\right)\right]+\frac{1}{Re}\left[\bar{\mu}\left(f_3''+\iota\beta f_2'-\alpha^2 f_3-\alpha\beta f_1\right)\right] \quad (48)$$

$$\bar{\rho}\left(\iota\left(\alpha\bar{u}-\omega\right)f_5+f_2\frac{d\bar{T}}{dy}\right)=\left(\gamma-1\right)M^2\iota(\alpha\bar{u}-\omega)\bar{p}\left(\frac{f_4}{\bar{\rho}}\right.$$
$$\left.+\frac{f_5}{\bar{T}}\right)-\frac{1}{RePr}\left(\bar{k}f_5\right)\left(\alpha^2\right.$$
$$\left.+\beta^2\right)+\frac{1}{RePr}\left(\bar{k}f_5''\right.$$
$$+2\frac{d\bar{k}}{d\bar{T}}\frac{d\bar{T}}{dy}f_5'+\frac{d^2\bar{k}}{d\bar{T}^2}\left(\frac{d\bar{T}}{dy}\right)^2f_5+$$
$$\left.\frac{d\bar{k}}{d\bar{T}}\frac{d^2\bar{T}}{dy^2}f_5\right)+\frac{\left(\gamma-1\right)M^2}{Re}\left\{2\bar{\mu}\frac{d\bar{u}}{dy}\right.$$
$$\left.\left(f_1'+f_2\iota\alpha\right)+\frac{d\bar{\mu}}{d\bar{T}}f_5\left(\frac{d\bar{u}}{dy}\right)^2\right\} \tag{49}$$