

A tour of singular perturbation methods for differential equations

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Inter Group Meeting, IITB
12th Feb, 2016

Topics in perturbation methods

- **Local Analysis**

- Near Ordinary points - *Taylor series*
- Near Regular singular points - *Frobenius method*
- Near Irregular points - *Method of dominant balance*

- **Global Analysis**

- Boundary layer theory
- WKBJ analysis
- Method of multiple scales

Topics in perturbation methods

- **Local Analysis**

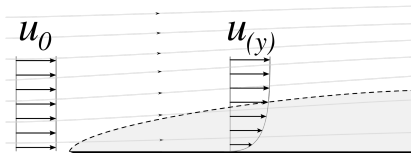
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- **Global Analysis**

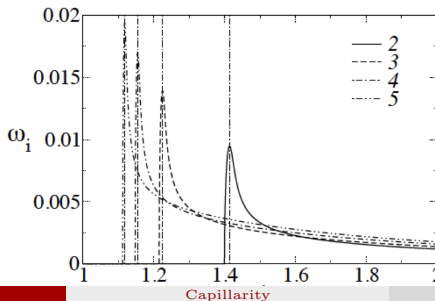
- **Boundary layer theory**
- WKBJ analysis
- Method of multiple scales

Some examples

Boundary layer over a flat plate: (Source: Wikipedia)



Critical layer singularity in hydrodynamic stability:



First order differential equation

Solve:

$$\epsilon y' + y = \sin x, \quad y(0) = 1 \quad \text{for } x \geq 0.$$

Exact solution:

$$y_{exact} = \frac{\sin x - \epsilon \cos x}{1 + \epsilon^2} + \frac{\epsilon}{1 + \epsilon^2} e^{-x/\epsilon}.$$

Perturbation solution:

- ① Set $\epsilon = 0 \implies y = \sin x$ *Outer solution*
Does not obey the condition at $x = 0$. Therefore this is a *Singular Perturbation Problem*.
- ② Let there be a region of size δ near $x = 0$ such that

$$x = \delta X, \quad y(x) = Y(X).$$

The first derivative becomes

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{1}{\epsilon} \frac{dY}{dX}.$$

First order differential equation - contd.

The differential equation becomes

$$\frac{\epsilon}{\delta} \frac{dY}{dX} + Y = \sin(\delta X), \quad \text{with} \quad Y(0) = 1.$$

To retain first derivative term at small ϵ , we choose

$$\delta = \epsilon.$$

We have a 'boundary layer' of size ϵ near $x = 0$.

$$\boxed{\frac{dY}{dX} + Y = \sin(\epsilon X), \quad \text{with} \quad Y(0) = 1.}$$

Solving with regular perturbation expansion, we write

$$Y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \dots \quad \text{and use} \quad \sin(\epsilon X) \approx \epsilon X - \frac{\epsilon^3 X^3}{3!} + \dots$$

$O(1)$ solution:

$$Y_0'(X) + Y_0(X) = 0, \quad \text{with} \quad Y_0(0) = 1.$$

$$\implies Y_0(X) = e^{-X}.$$

First order differential equation - contd.

$O(\epsilon)$ solution:

$$Y_1'(X) + Y_1(X) = X, \quad \text{with} \quad Y_1(0) = 0.$$

The solution of this equation gives

$$Y_1(X) = X - 1 + e^{-X}.$$

Putting it all together, the inner solution becomes

$$Y = e^{-X} + \epsilon(X - 1 + e^{-X}) + O(\epsilon^2).$$

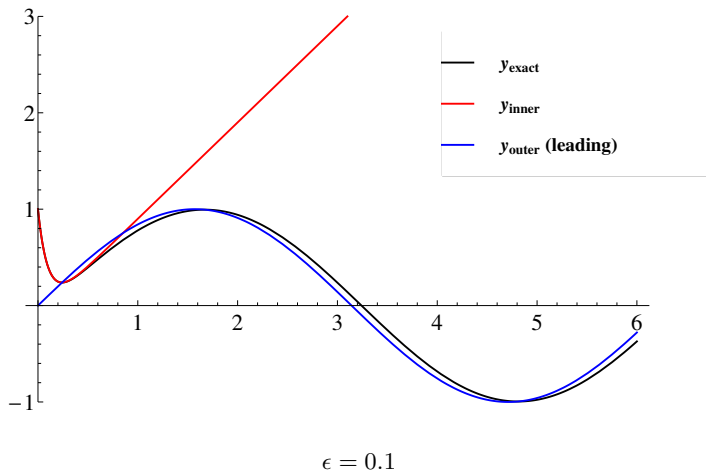
In terms of regular variable x , the inner solution becomes

$$y = e^{-x/\epsilon} + (x - \epsilon + \epsilon e^{-x/\epsilon}) + O(\epsilon^2).$$

This solution clearly satisfies the boundary condition $y(0) = 1$. Since $\epsilon \ll 1$, the $e^{-x/\epsilon}$ term decays rapidly causing the solution to have a large variation near $x = 0$. This rapid variation, confined to a $O(\epsilon)$ region is essential for the solution to satisfy the boundary condition at $x = 0$. We therefore call these types of problems as *boundary layer problems*.

First order differential equation - contd.

We now compare the exact solution obtained above with the perturbation solutions.



Second order singular boundary layer

Solve:

$$\epsilon y'' + y' = e^{-x}, \quad y(0) = 1, \quad y(1) = 1.$$

Let $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$

$O(1)$:

$$y'_0 = e^{-x} \implies y_0 = c_0 - e^{-x}. \quad \text{Outer solution}$$

To find c_0 , we need to use boundary condition at either $x = 0$ or $x = 1$. But which one?

Second order singular boundary layer

Let there be a boundary layer at $x = 0$:

$$x = \delta X, \quad \text{with } \delta \ll 1,$$

$$y(x) = Y(X).$$



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$$Y'' + Y' = \epsilon e^{-\epsilon X}$$

$$\text{Using } Y = Y_0(X) + \epsilon Y_1(X) + \dots,$$

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$$\text{Using } Y = Y_0(X) + \epsilon Y_1(X) + \dots,$$

$$Y_0'' + Y_0' = 0.$$

$$Y_0(X) = c_1 + c_2 e^{-X},$$

$$\text{with } Y_0(X = 0) = 1,$$

$$\implies Y_0(X) = e^{-X} + c_1 (1 - e^{-X})$$

This solution decays as $X \rightarrow \infty$, hence acceptable.

Second order singular boundary layer

Let there be a boundary layer at $x = 1$:

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$$\text{Using } Y = Y_0(X) + \epsilon Y_1(X) + \dots,$$

$$Y_0'' - Y_0' = 0.$$

$$Y_0(X) = c_2 e^X - c_1,$$

This solution grows exponentially as $X \rightarrow \infty$.

Second order singular boundary layer

Outer solution

$$y_0(x) = c_0 - e^{-x},$$

$$y_0(1) = 1,$$

$$\implies y_0(x) = 1 + e^{-1} - e^{-x}.$$

Inner solution

$$Y_0(X) = e^{-X} + c_1(1 - e^{-X}),$$

$$Y_0(0) = 1.$$



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Inner solution

$$Y_0(X) = e^{-X} + c_1(1 - e^{-X}),$$

$$Y_0(0) = 1.$$



To only parameter left undetermined is c_1 . We find this by matching inner and outer solutions.

van Dyke Matching principle:

$$\lim_{X \rightarrow \text{towards outer}} Y(X) = \lim_{x \rightarrow \text{inner}} y(x),$$

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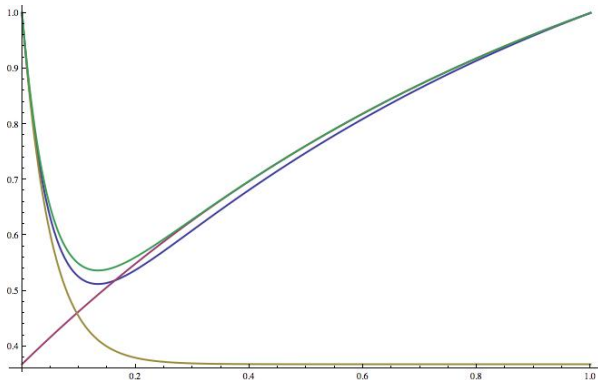
$$\lim_{X \rightarrow \text{towards outer}} Y(X) = \lim_{x \rightarrow \text{inner}} y(x),$$

$$\lim_{X \rightarrow \infty} Y(X) = \lim_{x \rightarrow 0} y(x)$$

Second order singular boundary layer

Composite (uniformly valid) solution:

$$\begin{aligned}y_{comp} &= y_{inner} + y_{outer} - y_{overlap}, \\&= Y_0(X) + y_0(x) - y_{overlap}, \\&= 1 - e^{-x} + e^{-x/\epsilon} - e^{-(1+x/\epsilon)} - 1/e\end{aligned}$$



Home work

Solve:

$$\epsilon y'' - x^2 y' - y = 0, \quad y(0) = 1, \quad y(1) = 1.$$

Home work

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There are boundary layers at both ends

Blasius boundary layers

Prandtl's boundary layer equations (first proposed in 1904)

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2},$$
$$\frac{\partial p}{\partial y} = 0.$$

Solved by his student Paul Heinrich Blasius in 1908:

Using $u = U f'(\eta)$, where $\eta = \frac{y}{\delta(x)} = y \left(\frac{U}{\nu x} \right)^{1/2}$,
we get,

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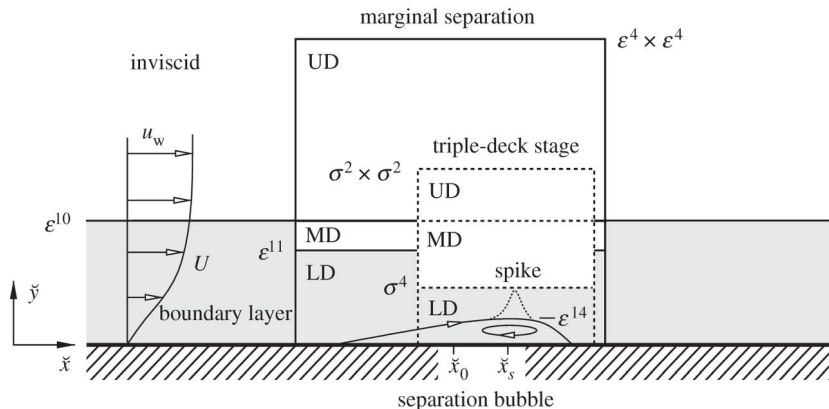
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we get,

$$f''' + \frac{1}{2} f f'' = 0.$$

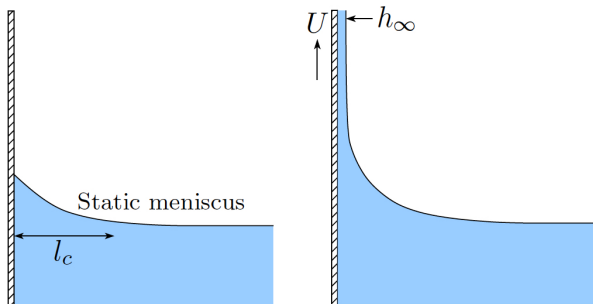
No scope for learning about “matching” in this problem. Problem of Matching arises in higher order boundary layer theory.

Higher order boundary layer structure near separation



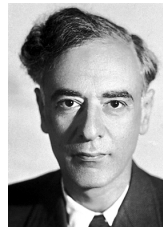
Source: Braun & Scheichl, Phil. Trans. Roy. Soc., 2014

Landau-Levich flow

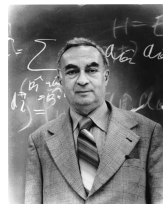


Landau-Levich law: $h_\infty = 0.9458 l_c Ca^{2/3}$ where $Ca = \left(\frac{\mu U}{\sigma}\right)$ and $l_c = \sqrt{\frac{\sigma}{\rho g}}$ is the capillary length

Landau & Levich, *Acta Physicochim.* (1942)



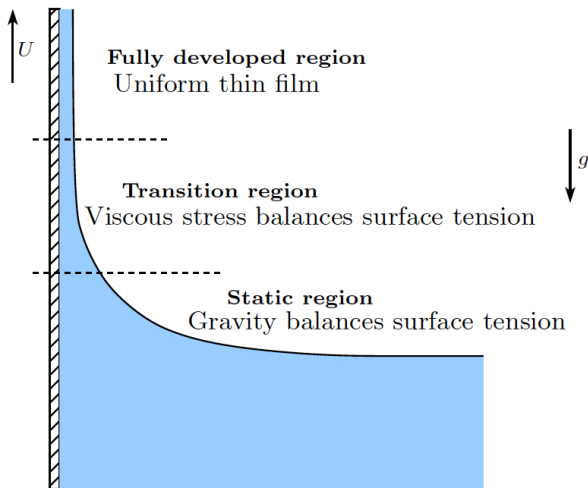
Lev Landau



Benjamin Levich

Landau-Levich law: $h_\infty = 0.9458 l_c Ca^{2/3}$

- Landau and Levich used the ideas of matched asymptotic expansions before their formal development in obtaining this result



Landau & Levich, *Acta Physicochim.* (1942)

Landau-Levich flow

$$u_x + v_y = 0,$$

$$CaRe(uu_x + vv_y) = -p_x + Ca\nabla^2 u - 1,$$

$$CaRe(uu_x + vv_y) = -p_x + Ca\nabla^2 v,$$

$$u = 1, v = 0 \quad \text{at} \quad y = 0.$$

At $y = h(x)$:

$$v - h_x u = 0,$$

$$-p + \frac{2Ca}{1 + h_x^2} \{u_x(h_x^2 - 1) - (u_y + v_x)h_x\} = \frac{h_{xx}}{(1 + h_x^2)^{3/2}},$$

$$-4u_x h_x + (u_y + v_x)(1 - h_x^2) = 0.$$

Dip-coating asymptotics

All lengths are non-dimensionalized by l_c and velocities by U .

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Transition region scaling:

$$(\bar{x}, \bar{y}) = \left(\frac{x + l}{Ca^{1/3}}, \frac{y}{Ca^{2/3}} \right),$$

$$(\bar{u}, \bar{v}) = \left(u, \frac{v}{Ca^{1/3}} \right),$$

$$\bar{p} = p, \quad \bar{h} = \frac{h}{Ca^{2/3}}.$$

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Transition region

$$\bar{p}_{\bar{x}} = \bar{u}_{\bar{y}}\bar{y},$$

$$\bar{p} + \underbrace{\bar{h}_{\bar{x}\bar{x}}}_{\text{Surface tension}} = 0.$$

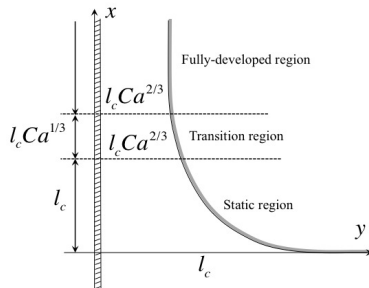
Matching the solution in the two regions, we get $h_{\infty}/l_c = 0.9458Ca^{2/3}$

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- Find an outer solution
- Find an inner solution
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When you follow two separate chains of thought, Watson, you will find some point of intersection which should approximate the truth.

– Sherlock Holmes, *The Disappearance of Lady Francis Carfax*,
Sir Arthur Conan Doyle