

Conservative formulation of shock-unsteadiness k - ϵ turbulence model

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Presented by

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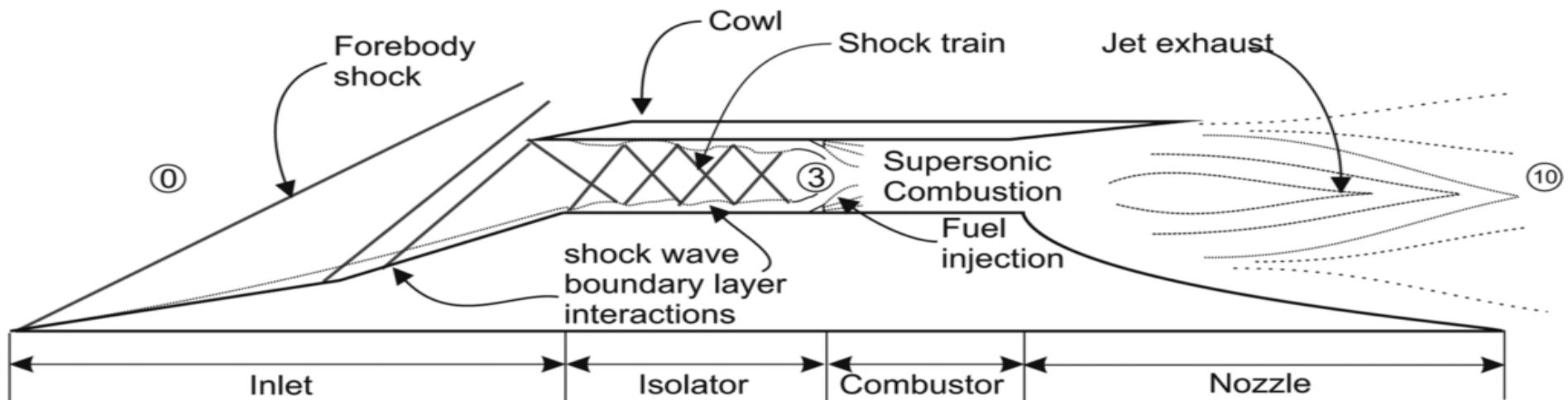


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Introduction

Numerical predictions of shock wave/turbulent boundary layer interactions (STBLI's) is important

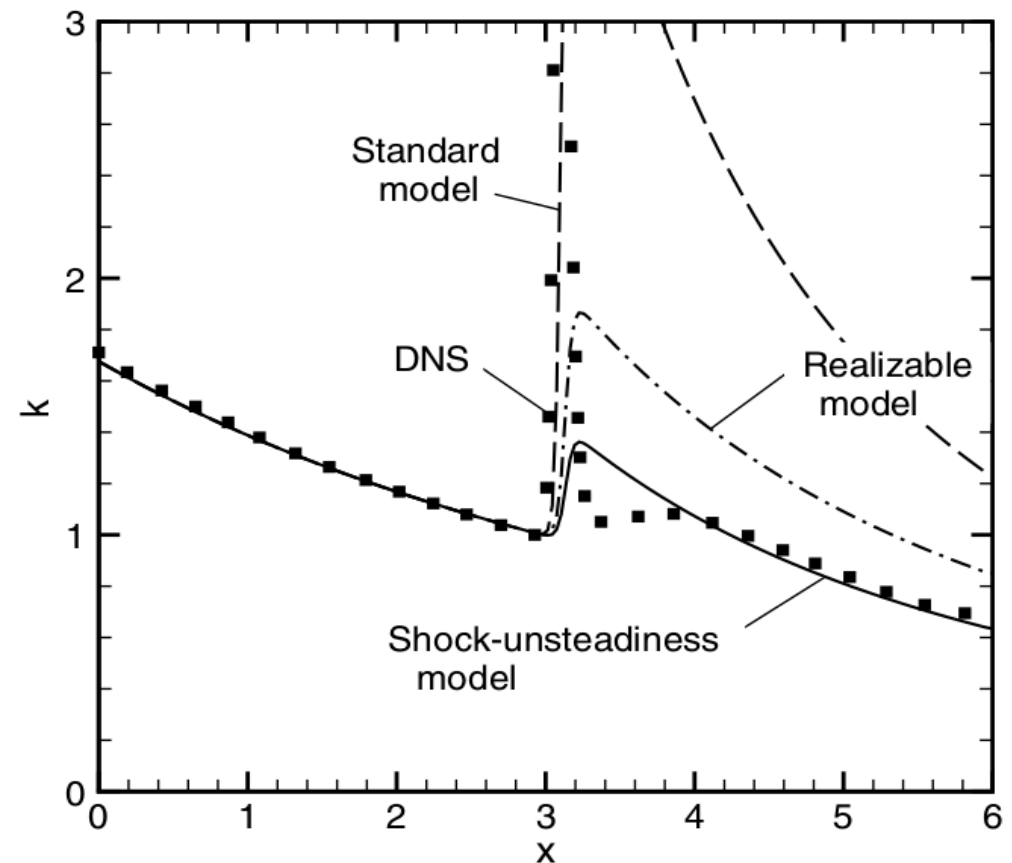
- **STBLI's :**
 - High speed flows
 - e.g. intake of scramjet engines, transonic wings, fins of missiles, cone-flare configurations etc.



Schematic of Scramjet engine

Sinha et al. 2003 shock-unsteadiness modification

- Turbulence gets amplified upon interaction with a shock wave.
- Conventional turbulence models overpredict turbulence kinetic energy (TKE) across a shock-wave.
- Shock-unsteadiness modification improves the performance of standard turbulence models.
- Amplification of TKE across a shock wave matches well with the DNS data.



Mach 1.5 canonical shock/turbulence interaction
(ref. Sinha and Balasridhar 2013)

Sinha et al. 2003 (Phys. Fluids) shock-unsteadiness k-ε model

- Turbulent kinetic energy equation :
$$\bar{\rho} \tilde{u} \frac{\partial k}{\partial x} = \boxed{-\frac{2}{3} \bar{\rho} k \frac{\partial \tilde{u}}{\partial x} + \frac{2}{3} b'_1 \bar{\rho} k \frac{\partial \tilde{u}}{\partial x}} - \bar{\rho} \epsilon_s$$

where $b'_1 = b_{1,\infty}(1 - e^{1-M_1})$ is the shock-unsteadiness damping parameter and $b_{1,\infty} = 0.4$

- Turbulent dissipation rate equation :
$$\bar{\rho} \tilde{u} \frac{\partial \epsilon}{\partial x} = \boxed{-\frac{2}{3} c_1 \bar{\rho} \epsilon \frac{\partial \tilde{u}}{\partial x}} - c_2 \frac{\bar{\rho} \epsilon^2}{k}$$

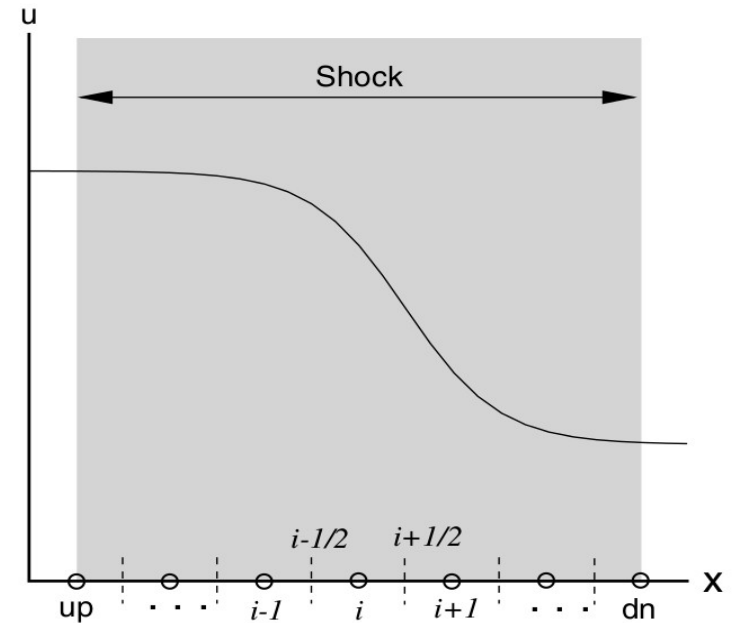
where $c_1 = 1.2 + 0.21M$ and $c_2 = 1.2$

Non-conservative source terms !!

Numerical error analysis at the shock wave (Sinha & Balasridhar 2013 AIAA J.)

- Finite difference representation of the non-conservative derivative terms do not form a telescoping series
- Large discretisation error at the shock wave
- Model equation :

$$\frac{\partial h}{\partial x} = \psi \quad \text{say} \quad \psi = -\frac{2}{3} c_0 \bar{\rho} k \frac{\partial \tilde{u}}{\partial x}$$



- Finite difference representation : $\psi \simeq -\frac{2}{3} c_0 \bar{\rho} k \left(\frac{\tilde{u}_{i+1} - \tilde{u}_{i-1}}{2\Delta x} \right)$
- Using Taylor's Series:

$$= -\frac{2}{3} c_0 \bar{\rho} k \left[\left(\frac{\partial \tilde{u}}{\partial x} \right)_i + \frac{\Delta x^2}{3} \left(\frac{\partial^3 \tilde{u}}{\partial x^3} \right)_i + \dots \right]$$

$$\frac{\partial \tilde{u}}{\partial x} \sim \frac{\Delta \tilde{u}}{\Delta x} \sim \frac{\Delta \tilde{u}}{\delta}$$

Non-conservative error in the shock region

$$-\frac{2}{3} c_0 \int_{x_{\text{up}}}^{x_{\text{dn}}} \bar{\rho} k \frac{\partial \tilde{u}}{\partial x} dx - \frac{2}{3} c_0 \frac{\Delta x^2}{3} \int_{x_{\text{up}}}^{x_{\text{dn}}} \bar{\rho} k \frac{\partial^3 \tilde{u}}{\partial x^3} dx - \dots$$

$$\frac{\partial^3 \tilde{u}}{\partial x^3} \sim \frac{\Delta \tilde{u}}{\delta^3}$$

Conservative formulation

- Conservative formulation to limit the error
- Analytical integration of the shock-unsteadiness k - ε equations :

$$\bar{\rho} \tilde{u} \frac{\partial k}{\partial x} = -\frac{2}{3} \bar{\rho} k \frac{\partial \tilde{u}}{\partial x} + \frac{2}{3} b'_1 \bar{\rho} k \frac{\partial \tilde{u}}{\partial x} - \bar{\rho} \epsilon_s \quad \text{and} \quad \bar{\rho} \tilde{u} \frac{\partial \epsilon}{\partial x} = -\frac{2}{3} c_1 \bar{\rho} \epsilon \frac{\partial \tilde{u}}{\partial x} - c_2 \frac{\bar{\rho} \epsilon^2}{k}$$

$$\frac{k_2}{k_1} = \left(\frac{\bar{\rho}_2}{\bar{\rho}_1} \right)^{\frac{2}{3} c_0} \quad \text{and} \quad \frac{\epsilon_2}{\epsilon_1} = \left(\frac{\bar{\rho}_2}{\bar{\rho}_1} \right)^{\frac{2}{3} c_1} \quad \text{where} \quad c_0 = 1 - b'_1$$

- Conserved quantites :

Exact

$$f = k \rho^{-\frac{2}{3} c_0} \quad \text{and} \quad g = \epsilon \rho^{-\frac{2}{3} c_1}$$

- Equations in conservative form :

$$\rho \tilde{u} \frac{\partial f}{\partial x} = -\rho g \rho^{\frac{2}{3}(c_1 - c_0)} \quad \rho \tilde{u} \frac{\partial g}{\partial x} = -c_2 \frac{\rho g^2}{f} \rho^{\frac{2}{3}(c_1 - c_0)}$$

Numerical error analysis of conservative derivative

$$\frac{\partial h}{\partial x} \simeq \frac{h_{i+1/2} - h_{i-1/2}}{\Delta x}$$

- Discretisation error : $[h_{\text{dn}} - h_{\text{up}}] + \frac{\Delta x^2}{24} \left[\left(\frac{\partial^2 h}{\partial x^2} \right)_{\text{dn}} - \left(\frac{\partial^2 h}{\partial x^2} \right)_{\text{up}} \right] + \dots$
- Telescoping effect

Results

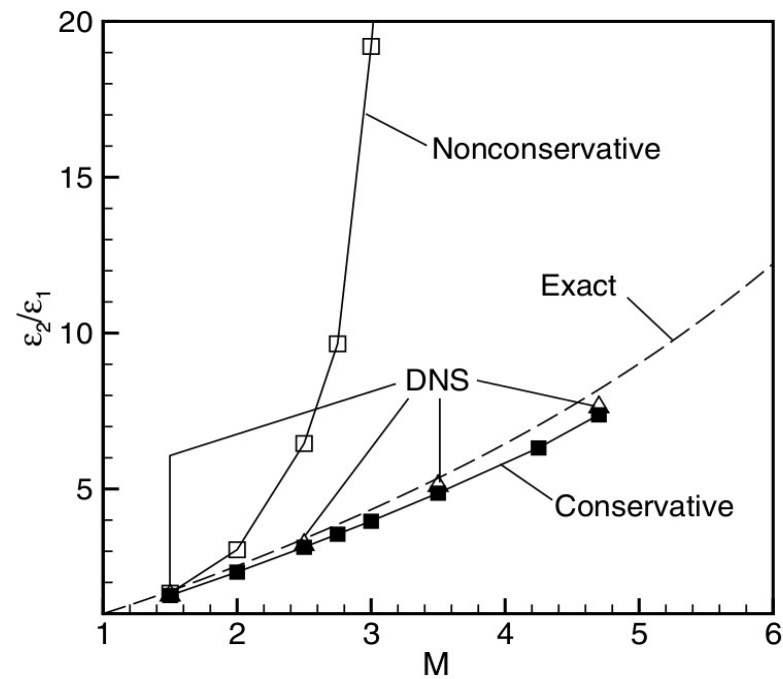
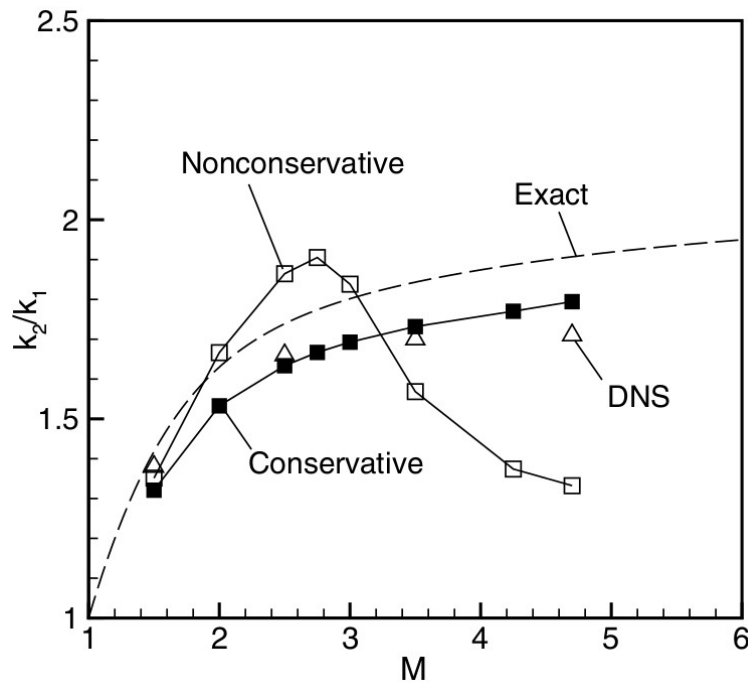


Fig. TKE and epsilon amplifications with upstream Mach number (ref. Sinha and Balasridhar 2013)

Grid Convergence

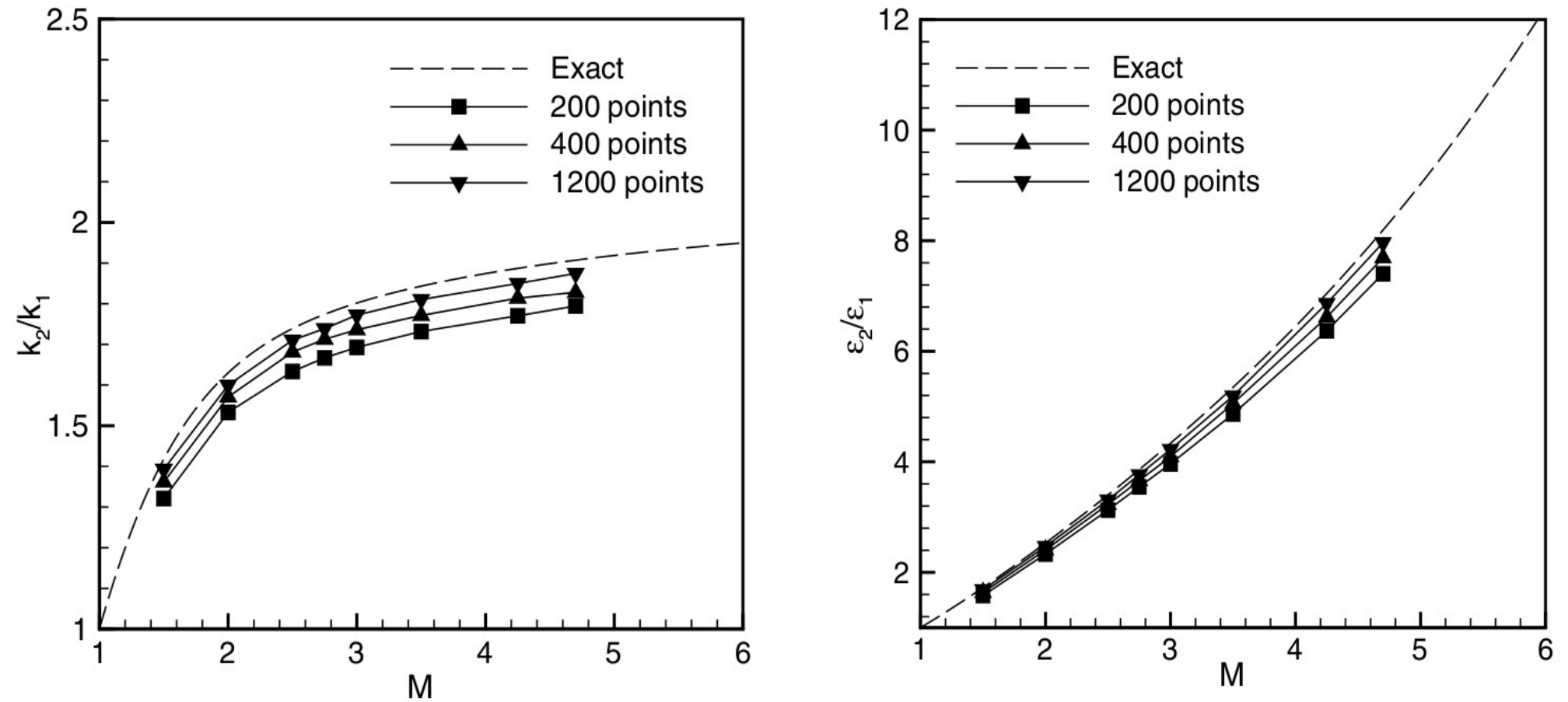


Fig. Grid Convergence for conservative formulation
(ref. Sinha and Balasridhar 2013)

Comparisons with DNS

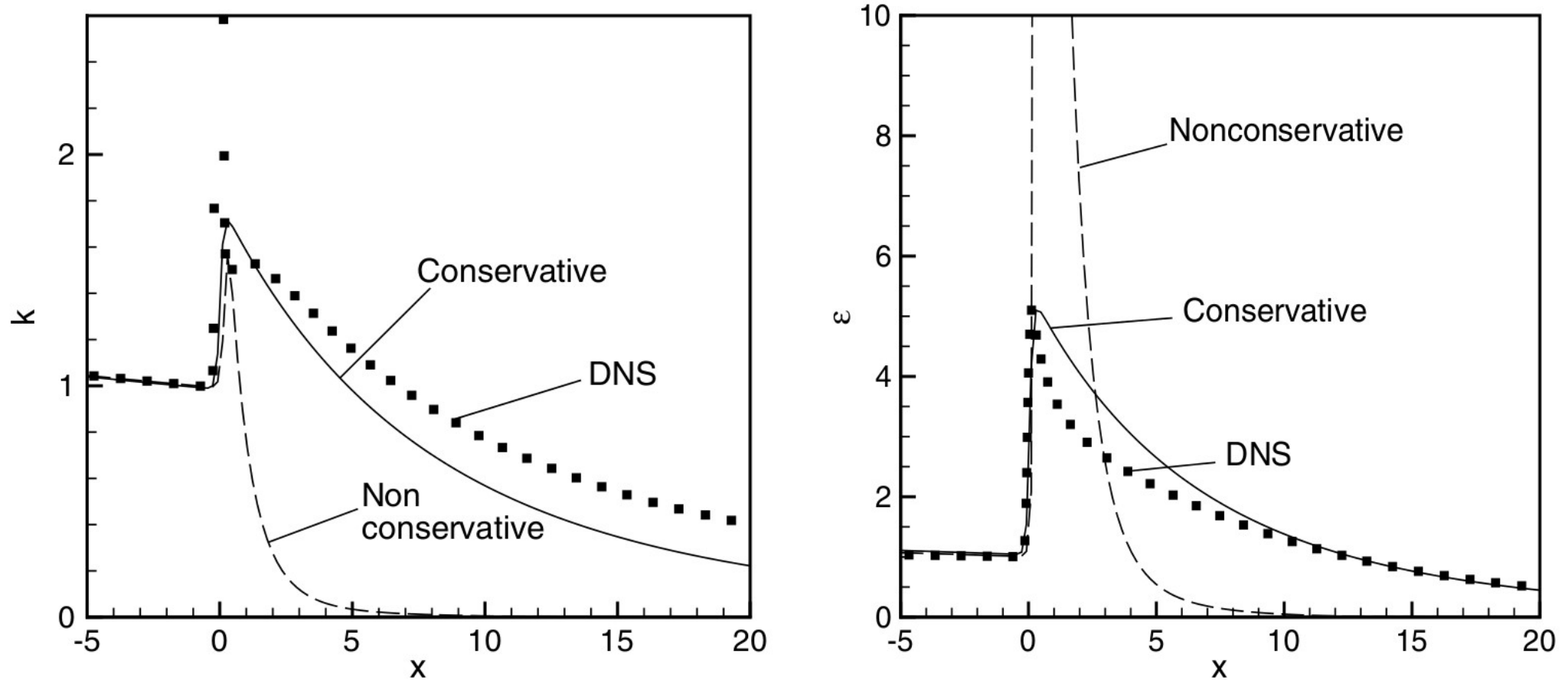


Fig. Streamwise evolutions of TKE and epsilon for $M = 3.5$
(ref. Sinha and Balasridhar 2013)

Advantages of Conservative formulation

- Limits large numerical error at the shock wave
- Similar to the non-conservative form
- Easy implementation in the CFD code

Thank You