

Effect of transport property variation on the stability of high-speed flows

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by

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Approval

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Abstract

Accurate prediction of laminar to turbulent transition in high-speed flows is a challenging task. In a compressible flow, changes in pressure, temperature and gas composition give rise to large non-monotonic variations in the transport properties of the gas. Changes in these properties can affect the stability characteristics of a high-Mach number flow in a different way due to the presence of additional instability modes compared to an incompressible flow. In this work, we investigate the effect of variations in the transport properties on flow stability, by constructing different model problems with varying viscosity and thermal conductivity. The main objective of formulating these model problems is to isolate the effect of individual and coupled variations of transport properties.

To study the stability behavior of a wall-bounded flow, we consider a compressible plane Couette flow of perfect gas. The asymptotic characteristics of the disturbances are examined using the linear stability theory, and short-term energy growth has been investigated with the help of non-modal analysis or the transient growth. We observe that an increase in the thermal conductivity can destabilize the Couette flow significantly. Stratification in viscosity is found to have a dominant stabilizing role. In a flow, where both viscosity and conductivity vary simultaneously, we notice that the viscosity stratification dictates the stability aspects. These outcomes of our work are valid at a finite time as well as in the asymptotic limit, and also for a range of free-stream parameters.

A detailed energy analysis is carried out to study the effect of individual transport properties on various constituents of total perturbation energy. A balance between the decay and redistribution of energy transferred from the mean flow determines the optimal energy growth of disturbances. The physical mechanisms leading to the maximum transient energy amplification for each of the model configurations are also studied. The key mechanism for energy growth in a finite time is found to be the lift-up process, in the majority of flow cases considered. A

combination of lift-up and Orr-mechanism decide the non-modal energy growth in the presence of both viscosity and conductivity stratification.

We find that the effects of viscosity and conductivity variation can be quantified by a single non-dimensional parameter, which is the ratio of momentum diffusivity to thermal diffusivity or the Prandtl number of fluid. Increasing the viscosity or decreasing the thermal conductivity value stabilizes the Couette flow, which is the same as increasing the Prandtl number and vice versa. We extend the study to the high-speed boundary flow over a flat plate, and a series of numerical experiments are carried out to investigate the effect of a change in the Prandtl number on temporal growth rates. The stability diagrams plotted for a range of Reynolds number and wavenumber show a destabilizing role for increasing the Prandtl number, leading to larger regions of instability and increased growth rates. This also results in a significant reduction in the critical Reynolds number, especially at intermediate Mach numbers. Two types of branching patterns are observed depending on the Prandtl number, where either the fast or the slow mode can be destabilized due to mode-synchronization. The effect of wall-cooling on the second mode is also examined at different Prandtl numbers. We notice that cooling increases the amplification rate and shifts the peak growth rate to a higher wavenumber value.

Keywords: Transition, compressibility, transport properties, Prandtl number, viscosity stratification, thermal conductivity stratification, linear stability theory, transient growth, Couette flow, boundary layer, branching pattern, wall-cooling.

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Nomenclature

x, y, z	Cartesian coordinates
u, v, w	Velocity components in x, y and z direction
γ	Ratio of specific heats
ρ	Density
T	Temperature
M	Mach number
μ	Dynamic Viscosity
κ	Thermal conductivity
Re	Reynolds number
Re_{cr}	Critical Reynolds number
Pr	Prandtl number
c	Phase speed
ω	Wave frequency
α	Wave number component in x direction
β	Wave number component in z direction
α_{cr}	Critical wavenumber

Conventions

*	Dimensional quantities
$\bar{A}$	Mean flow quantities
avg	Averaged quantities
$\tilde{A}$	Fluctuation quantities
$\hat{A}$	Complex amplitude of the fluctuations
cc	Complex conjugate

Chapter 1

Introduction

Hypersonic vehicles fly at an extreme range of stagnation enthalpy, and the aero-thermodynamics associated with them involve strong shock waves, viscous shock layers and non-equilibrium thermo-chemistry. These multiple processes can influence laminar to turbulent transition in different ways. A turbulent boundary layer generates much higher friction and heat transfer, by factors of four or higher to the vehicle surface compared to a laminar boundary layer. The prediction of laminar to turbulent transition is an important concern since it decides the amount of aero-heating and skin friction drag, which in turn affect the vehicle weight and material, range, and payload capacity. For avoiding an overly conservative design margin for the thermal protection system it is crucial to predict the location and stream-wise extent of transition accurately.

1.1 Factors affecting transition

The process of transition in a high-speed flow can be affected by a wide range of factors [1]-[5]. For example, receptivity to free-stream disturbance, roughness, the presence of surface curvature and nose bluntness can delay or augment the transition process. In addition, boundary conditions imposed on the vehicle surface, such as cold/adiabatic, smooth/rough, catalytic/non-catalytic can delay or augment the transition process. The variation in thermodynamic and transport properties of the gas can also have a major influence on flow stability [6]- [7]. In the context of drag reduction, the role of transport property variation, mainly that of viscosity stratification, has been studied widely for incompressible as well as non-Newtonian flows and

is well documented in Ref. [8]- [18].

To demonstrate the destabilizing role of viscosity stratification, Yih [8] considers a plane Couette-Poiseuille flow of two superposed layers of fluids with different viscosities. He finds that if the viscosity varies between the layers, and the depth ratio and the viscosity ratio are within certain ranges, both plane Poiseuille and Couette flow can be unstable at any Reynolds number. In the case of incompressible boundary layers, Wall and Wilson [9] examine the effect of different viscosity models on flow stability and notice that a non-uniform decrease in the viscosity across the boundary layer stabilizes the flow, and the extent of stabilization increases as the Prandtl number is increased. To investigate the role of variation of thermal conductivity on flow stability, Sorokin [10] studies a vertical plane fluid layer with thermal conductivity depending linearly on temperature. He observes that relatively small changes in the thermal conductivity caused by a variation in temperature results in flow destabilization.

The crucial role of continuous viscosity stratification at the critical layer, where the phase velocity of the disturbance coincides with the mean flow velocity, is first realized by Craik [11] for a plane Couette flow. Ranganathan and Govindarajan [12] study a channel flow of two fluids of different viscosities with a mixed layer in between and find that an overlap of the critical layer with the layer of viscosity stratification leads to an order of magnitude stabilization of the flow. This stabilization is attributed to the reduction in energy intake from the mean flow to the disturbance in Ref. [13]. Govindarajan et al. [14] also observe that the exact form of the viscosity profile is immaterial; any monotonic continuous profile of viscosity in the thin mixed layer gives the same answer. In Ref. [15], Govindarajan finds that for a channel flow with two fluids of high mass diffusivity, a new mode of instability (overlap mode) appears, when the critical layer of the dominant disturbance overlaps the viscosity stratified layer.

The role of viscosity stratification on the transient energy growth of the disturbance has also been studied for incompressible flows. Chikkadi et al. [16] notice that transient growth of disturbances along with the streamwise vortices are practically unaffected by the viscosity stratification. Sameen and Govindarajan [17] also observe a similar effect of viscosity stratification, but they notice that increasing the relative magnitudes of momentum to thermal diffusivity, i.e., the Prandtl number, transient growth increases dramatically. Nouar et al. [18] revisit a few previous studies related to the stability of viscosity stratified channel flow and mention that for Carreau fluids, the transition is effectively postponed when a viscosity contrast is produced in

the critical layer.

In the case of compressible flows, it is well known that the coupling between the momentum and energy equations leads to the existence of multiple instability modes, as opposed to the single instability mode (Tollmien-Schlichting mode) of incompressible flows. Moreover, the same factors which act to stabilize the incompressible flow can have a destabilizing effect for compressible flows. For example, wall cooling is found to stabilize the first instability mode, but it strongly destabilizes the second mode, which is dominant at high Mach numbers.

1.2 Motivation

In the case of high-speed flows, large changes in pressure, density, temperature and gas composition, in the presence of chemical reactions, can lead to a large variation in the transport properties of the gas. Sutherland's formula with a fixed value of Prandtl number is standardly used to compute the transport properties, such that variations in viscosity and thermal conductivity are coupled. More realistic estimates of transport properties, especially in high-Mach number applications, can be obtained using Blottner curve-fits with mixing rule or collision cross-section based formulations. Viscosity and conductivity data for air as a function of pressure and temperature (Fig. 1.1) shows large non-monotonic variation, which leads to a significant change in the effective Prandtl number. Further, the variation of transport properties and the resulting effect on flow instabilities, are dependent on the specific transport model used. The problem is exacerbated by the fact that stability results are often very sensitive to minute changes in the flow properties. This is a major disadvantage in studying the stability characteristics of high-speed flows, especially in the absence of a universally accepted transport model.

Stability studies of high-speed flows have reported that transport property effects can dominate over other thermodynamic and chemical reaction rate variations. Lyttle and Reed [7] consider a spherically blunted right circular cone at a Mach number of 13.5, and investigate the sensitivity of two different transport models (Stuckert and Blottner) on the second mode growth rate. They find that the peak growth rate of the second mode changes by around 8%, due to a 10% and 25% change in the viscosity and thermal conductivity values respectively. Franko et al. [6] consider a flat plate geometry at a Mach number of 10 and investigate the effect of using different chemistry models for thermodynamic, transport and chemical reactions on boundary

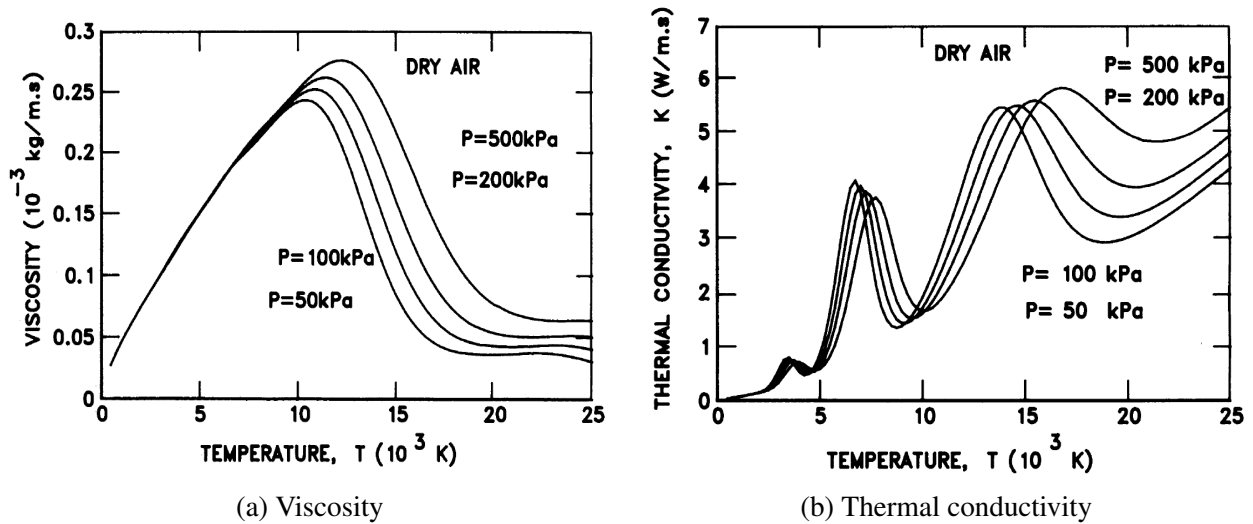


Figure 1.1: Variation of viscosity and thermal conductivity with pressure and temperature for equilibrium air [19].

layer stability. They report a 20% difference in the peak growth of the second mode, between the two transport models (Blottner and Chapman-Enskog method).

A significant amount of work has also been carried out to investigate the physical effect of transport properties on the stability of compressible flows. Using linear stability theory, Hu and Zhong [20] study the effect of viscosity on the first even and odd acoustic modes, by comparing the viscous results at finite Reynolds numbers with the inviscid results of Duck et al. [21]. They observe that viscosity can destabilize these modes for certain combinations of Reynolds number and wave number. To study the effect of stratification in viscosity, Malik et al. [22] compare their work on uniform viscosity Couette flow, with the stratified viscosity results of Hu and Zhong [20]. They notice that the gradient in viscosity has a profound stabilizing effect and it increases the critical Reynolds number of the flow significantly. However, a constant Prandtl number is assumed in these studies, which implies that viscosity and thermal conductivity are proportional and they vary together. This is often not the case in high-Mach number flows, where the transport properties have large independent variations with temperature and pressure (see Fig. 1.1). The current work investigates the effect of such variations in conductivity and viscosity on the flow stability at high Mach numbers.

1.3 Objective

The objective of our work is to understand separately the effects of viscosity, conductivity and their stratification on the stability of high-speed flows. In realistic flows, the variations of dif-

ferent transport properties are usually related to other effects of chemical reactions and internal energy excitation. The effect on stability is therefore due to a combination of multiple effects and it is difficult to know which one is responsible for a particular trend. There can also be competing effects, which can change from one flow configuration to another. In this report, we perform controlled studies by varying the individual transport properties to bring out the underlying physical effects. Combined and a coupled variation of viscosity and conductivity are also considered to understand how they come together. We suppress high-temperature effects, like chemical reactions and vibrational relaxation, and assume a perfect gas flow, so as to focus on transport property variations and their effect on flow stability. The effect of non-equilibrium thermo-chemistry on flow stability has been studied earlier, for example, Malik & Anderson [23], Hudson et al. [24] and others.

In this work, we have chosen a Couette flow to study the stability of stratified viscosity and conductivity flows, and compare it with that of flows with uniform transport properties. This flow has the advantage that geometry effects are kept to a minimum, and the shear is constant. Indeed, the stability of stratified flows in simple geometries, like Couette and Poiseuille, has been studied extensively in the literature since the classical work of Yih [8]. A reference flow is taken in which viscosity and conductivity are held constant at their respective reference values (typically, at the top-wall reference temperature). Two other flows are also considered – one with viscosity varying with temperature across the shear layer, keeping conductivity constant; and the other with uniform viscosity, but conductivity varying with temperature. We consider one more model problem where both viscosity and conductivity vary together in a coupled way, and the stability results for this case are compared with those mentioned above as well as the reference flow with uniform transport properties.

We perform a linear stability analysis to investigate the most unstable modes for all the model problems considered and how they depend on Mach, Reynolds, and disturbance wave numbers. The dominant growth rates and critical Reynolds numbers are compared to isolate the effects of viscosity and conductivity variation in the flow. Linear stability analysis is useful where the transition process begins with the laminar flow going linearly unstable to disturbance modes for a small range of wave numbers. However, in some cases, the eigenspectrum may consist of decaying modes alone, but the disturbance energy can grow transiently to very high values and enable nonlinearities to emerge. Therefore we have also conducted a non-modal

analysis to study the effect of variation of transport properties on the transient energy growth of disturbances. This will help us to identify any unstable region if exist, that was predicted as stable by the linear stability theory.

We note that the model problems presented above are theoretical constructs, specifically designed for the purpose outlined above. In real life, high-speed flows have myriad competing effects, and there is no way in an experiment to isolate each effect independently. Numerical experiments of the kind presented in this report can prove to be a valuable tool and possibly the only way to study and understand underlying physical effects. We also caution the reader that the present results are for the simplest of shear flows, namely a Couette flow and not trivially extendable to other geometries. However, given that the transport properties in a realistic high-Mach number flow depend on temperature, pressure, and composition in a non-trivial way, our findings of the individual effects of viscosity and conductivity could be of value to those studying more complex situations. We hope that these findings will motivate those working on more realistic compressible flows to examine whether transport property stratification can indeed be a source of destabilization.

This thesis is organized as follows. Chapter 2 deals with the basic instability mechanisms leading to the transition in high-speed as well as in incompressible flows at a low disturbance environment. The model problems and the stability approach are discussed in chapter 3. The numerical method adapted for solving the stability equations is also presented here. Chapter 4 deals with the reference flow. We describe the governing equations corresponding to this base flow and a stability analysis is also performed. The effect of variation in mean conductivity and its stratification are explored in the next chapter. To analyze the role of individual energy transfer terms, we also present a non-modal energy budget for the total perturbation energy in this chapter. In chapter 6, we discuss the role of viscosity and its stratification on the short term as well as the asymptotic growth of the disturbance. Further, we examine the combined effect of viscosity and conductivity on the stability of a Couette flow, which is presented in the next chapter. The physical mechanisms leading to the short-term energy growth of the perturbations are investigated in chapter 8. We also study the effect of variation of Prandtl number on the linear stability of a high-speed boundary flow, and the results are documented in chapter 9. The main outcomes of this study are summarized in chapter 10.

Chapter 2

Background literature

The laminar to turbulent transition is assumed to occur as a consequence of nonlinear feedback of the laminar boundary layer to diverse environmental disturbances. When these disturbances are small, the transition is attributed to the instabilities of a boundary layer and the transition process can be described by the linear stability theory. The various instability modes amplify and interact with one another resulting in the breakdown of the laminar flow. However, disturbances with higher amplitude can directly give rise to nonlinear instabilities leading to the turbulence. This report focuses on the transition process in a low-disturbance environment due to its practical and fundamental importance to the aerospace community. In this chapter, we review a few relevant theoretical and experimental works related to flow instability in low-speed as well as high-speed flows.

The physics underlying laminar to turbulent transition in a high-speed flow indicates that it is a multifold process. Depending on various mean flow parameters and disturbance environment, the transition can evolve in multiple ways. In a low-disturbance environment, modal growth leads to the transition of the laminar flow and it is also the most extensively reviewed amplification mechanism. Initially, the normal modes get excited by the receptivity process and then amplify when traveling downstream (linear eigenmode growth). Consequently, the exponential growth of instability waves leads to the nonlinear breakdown to turbulence. Linear stability theory (LST) has been extensively used to identify the unstable modes and predict their exponential growth or decay rate.

The earliest analytical works trying to explain the observed instabilities in a flow was based on the analysis of the inviscid problem, and it dates back to the nineteenth century. The pi-

oneering works from Rayleigh [25], [26], Tollmien [27] and Lin [28] lay the foundation of incompressible flow stability. Linear stability theory was introduced by Tollmien in 1931 [29], and for the first time calculated a meaningful critical Reynolds number. Schlichting extended Tollmien's work and computed the amplification rate of the most unstable mode as a function of the Reynolds number in a flat plate boundary layer [30], [31]. The vibrating ribbon experiment conducted in an incompressible boundary layer by Schubauer and Skramstad [32] validates the linear stability theory and provides the proof of the existence of Tollmien-Schlichting waves.

2.1 Inviscid instability: Incompressible flow

The initial works on the stability of parallel flow of inviscid fluid were carried out by Helmholtz [33], Kelvin [34], and Rayleigh [35]. A significant amount of work on the instability of inviscid flows especially, on the inflectional instability is available in the papers by Rayleigh ([26]-[39]). Inflectional instability implies that if an inviscid flow is unstable, then the mean flow velocity profile must have an inflection point somewhere in the domain, but the converse is not necessarily true. Rayleigh provided this necessary criterion for inviscid instability and to confirm the existence of an unstable disturbance we need another condition. Fjortoft came up with the sufficient condition to identify a velocity profile to be unstable or stable in the inviscid limit. The details of the two theorems are provided below.

2.1.1 Rayleigh inflection point criterion

To prove the inflection point theorem, let's consider a two-dimensional steady parallel flow, which is of the form $\vec{U} = \bar{U} \hat{i}$ (here, $\hat{i}$ denotes a unit vector in the streamwise direction). In this work, the dimensional quantities are denoted by an asterisk, or a star; whereas the corresponding non-dimensional quantities are unstarred. The flow is bounded by two planes $y = y_1$ and $y = y_2$, which may be either free or rigid. It is also assumed that viscosity only establishes the mean flow and it does not have any effect on the disturbances. Also, in the incompressible theory, a two-dimensional disturbance is known to be the most unstable at any Reynolds number (which is no longer true above about a Mach number of 1), three-dimensional waves are ignored in the current stability analysis. With these assumptions, Rayleigh's stability equation can be derived as,

$$D^2\hat{\psi} - \alpha^2\hat{\psi} - \frac{D^2\bar{U}}{\bar{U} - c}\hat{\psi} = 0. \quad (2.1)$$

Here, $\hat{\psi}$ denotes the complex amplitude of the disturbance stream function, and D stands for the derivative with respect to the wall-normal direction y . The component of the wavenumber vector ($\vec{\kappa}$) in the streamwise direction is represented by α . Since only two-dimensional disturbances are considered, the spanwise component of the wavenumber vector is zero ($\beta = 0$), and the total wavenumber is equal to the streamwise component ($\kappa = \alpha$).

Rayleigh considered a temporal problem (disturbance grows only in time), which implies that the disturbance phase speed is complex ($c = c_r + i c_i$), whereas the wavenumber is real. In addition, a non-neutral perturbation is assumed ($c_i \neq 0$). Since the Rayleigh stability equation is invariant under complex conjugation, hence if there exists a damped stable wave ($c_i < 0$), there will also be an amplified unstable wave ($c_i > 0$). Therefore, the condition for stability is that the phase speed is real, and for instability c is complex. Now, multiplying Eq. 2.1 by the complex conjugate of $\hat{\psi}$ ($= \hat{\psi}^{cc}$), and integrating between the boundaries y_1 to y_2 , we obtain,

$$\begin{aligned} & \int_{y_1}^{y_2} \hat{\psi}^{cc} D^2\hat{\psi} dy - \int_{y_1}^{y_2} \alpha^2 \hat{\psi}^{cc} \hat{\psi} dy - \int_{y_1}^{y_2} \frac{D^2\bar{U}}{\bar{U} - c} \hat{\psi}^{cc} \hat{\psi} dy = 0 \\ \Rightarrow & \hat{\psi}^{cc} \int_{y_1}^{y_2} D^2\hat{\psi} dy - \int_{y_1}^{y_2} \left(D\hat{\psi}^{cc} \int_{y_1}^{y_2} D^2\hat{\psi} dy \right) dy - \int_{y_1}^{y_2} \alpha^2 |\hat{\psi}|^2 dy - \int_{y_1}^{y_2} \frac{D^2\bar{U}}{\bar{U} - c} |\hat{\psi}|^2 dy = 0 \\ \Rightarrow & \left[\hat{\psi}^{cc} D\hat{\psi} \right]_{y_1}^{y_2} - \int_{y_1}^{y_2} |D\hat{\psi}|^2 dy - \int_{y_1}^{y_2} \alpha^2 |\hat{\psi}|^2 dy - \int_{y_1}^{y_2} \frac{D^2\bar{U}}{\bar{U} - c} |\hat{\psi}|^2 dy = 0 \\ \Rightarrow & \int_{y_1}^{y_2} \left(|D\hat{\psi}|^2 + \alpha^2 |\hat{\psi}|^2 \right) dy + \int_{y_1}^{y_2} \frac{D^2\bar{U}}{\bar{U} - c} |\hat{\psi}|^2 dy = 0. \end{aligned} \quad (2.2)$$

The proper boundary condition for the Rayleigh equation depends on the flow configuration. For example, boundary layers are with one end rigid, while the other boundary extends to infinity (free boundary). Whereas in the case of mixing layers, jets and wakes (unbounded flows) both the ends span to infinity. On a free boundary, the pressure is required to be constant. Here, we have considered a flow with two rigid boundaries at both the ends (y_1 and y_2), hence the impermeability condition is imposed ($\hat{\psi} = 0$), since the wall-normal velocity must vanish at the walls (inviscid flow).

The first integral in Eq. 2.2 is positive definite and real, and since the phase speed of the disturbance is complex, this makes the second integral complex-valued. Multiplying the numerator and denominator of the second term by $(\bar{U} - c^{cc})$, where, c^{cc} denotes the complex conjugate of phase speed, we obtain,

$$\begin{aligned}
 & \int_{y_1}^{y_2} \left(|D\hat{\psi}|^2 + \alpha^2 |\hat{\psi}|^2 \right) dy + \int_{y_1}^{y_2} \frac{(\bar{U} - c^{cc}) D^2 \bar{U}}{(\bar{U} - c)(\bar{U} - c^{cc})} |\hat{\psi}|^2 dy = 0 \\
 & \Rightarrow \int_{y_1}^{y_2} \left(|D\hat{\psi}|^2 + \alpha^2 |\hat{\psi}|^2 \right) dy + \int_{y_1}^{y_2} \frac{(\bar{U} - c_r + ic_i) D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy = 0 \\
 & \Rightarrow \int_{y_1}^{y_2} \left(|D\hat{\psi}|^2 + \alpha^2 |\hat{\psi}|^2 \right) dy + \int_{y_1}^{y_2} \frac{(\bar{U} - c_r) D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy + ic_i \int_{y_1}^{y_2} \frac{D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy = 0.
 \end{aligned} \tag{2.3}$$

The first two terms in the above equation are real, hence the imaginary term

$$c_i \int_{y_1}^{y_2} \frac{D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy = 0. \tag{2.4}$$

Both $|\hat{\psi}|^2$ and $|\bar{U} - c|^2$ are non-negative and c_i is non-zero. This implies that if the integral has to be zero, $D^2 \bar{U}$ needs to change the sign. In other words, there should be at least one inflection point in the mean flow velocity profile.

2.1.2 Fjortoft's theorem

In 1950, the Swedish meteorologist Fjortoft [40] provided a further necessary condition for inviscid instability, which states that there should be a maximum of vorticity for instability, in addition to the presence of a point of inflection in the flow. To prove this, we have to multiply Eq. 2.4 with $(\bar{U}_I - c_r)$, where $\bar{U}_I$ corresponds to the mean flow velocity at the point of inflection. This gives,

$$(\bar{U}_I - c_r) c_i \int_{y_1}^{y_2} \frac{D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy = 0$$

$$\Rightarrow (\bar{U}_I - c_r) \int_{y_1}^{y_2} \frac{D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy = 0, \quad (2.5)$$

which is valid only for a non-neutral disturbance ($c_i \neq 0$). Also equating the real parts of Eq. 2.3 to zero, we get,

$$\begin{aligned} & \int_{y_1}^{y_2} \left(|D\hat{\psi}|^2 + \alpha^2 |\hat{\psi}|^2 \right) dy + \int_{y_1}^{y_2} \frac{(\bar{U} - c_r) D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy = 0 \\ \Rightarrow & \int_{y_1}^{y_2} \frac{(\bar{U} - c_r) D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy = - \int_{y_1}^{y_2} \left(|D\hat{\psi}|^2 + \alpha^2 |\hat{\psi}|^2 \right) dy, \end{aligned}$$

which implies,

$$\int_{y_1}^{y_2} \frac{(\bar{U} - c_r) D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy < 0. \quad (2.6)$$

Subtracting Eq. 2.5 from 2.6, we obtain

$$\int_{y_1}^{y_2} \frac{(\bar{U} - \bar{U}_I) D^2 \bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy < 0.$$

For the integral on the above inequality to be negative, the expression $(\bar{U} - \bar{U}_I) D^2 \bar{U}$ must be negative somewhere in the flow domain. However, converse is not necessarily true, that means if $(\bar{U} - \bar{U}_I) D^2 \bar{U} < 0$ in any location in the flow, then the flow may not be unstable.

While Rayleigh's theorem states that the mean vorticity ($D\bar{U}$) must have a local maximum or minimum, Fjortoft's theorem provides a much stronger condition for inviscid instability. It asks for a local maximum in the mean vorticity profile somewhere within the domain, not at the boundaries. Therefore flows in jets, mixing layers, wakes and boundary layers with an adverse pressure gradient are potentially unstable, whereas Poiseuille or plane Couette flow or boundary layers with zero or favorable pressure gradients are stable due to the absence of an inflection point in the inviscid limit (see Fig. 2.1).

To demonstrate the instability criteria of Rayleigh and Fjortoft, we have used two velocity profiles: hyperbolic-sine ($\sinh y$), and the other one is hyperbolic-tangent shear layer ($\tanh y$) (see Fig. 2.2). Both the velocity distributions fulfill Rayleigh's inviscid instability condition since they have a point of inflection at $y = 0$. However, only the velocity profile in Fig. 2.2b satisfy Fjortoft's sufficient condition ($(\bar{U} - \bar{U}_I) D^2 \bar{U} \leq 0$) in addition to Rayleigh's inflection

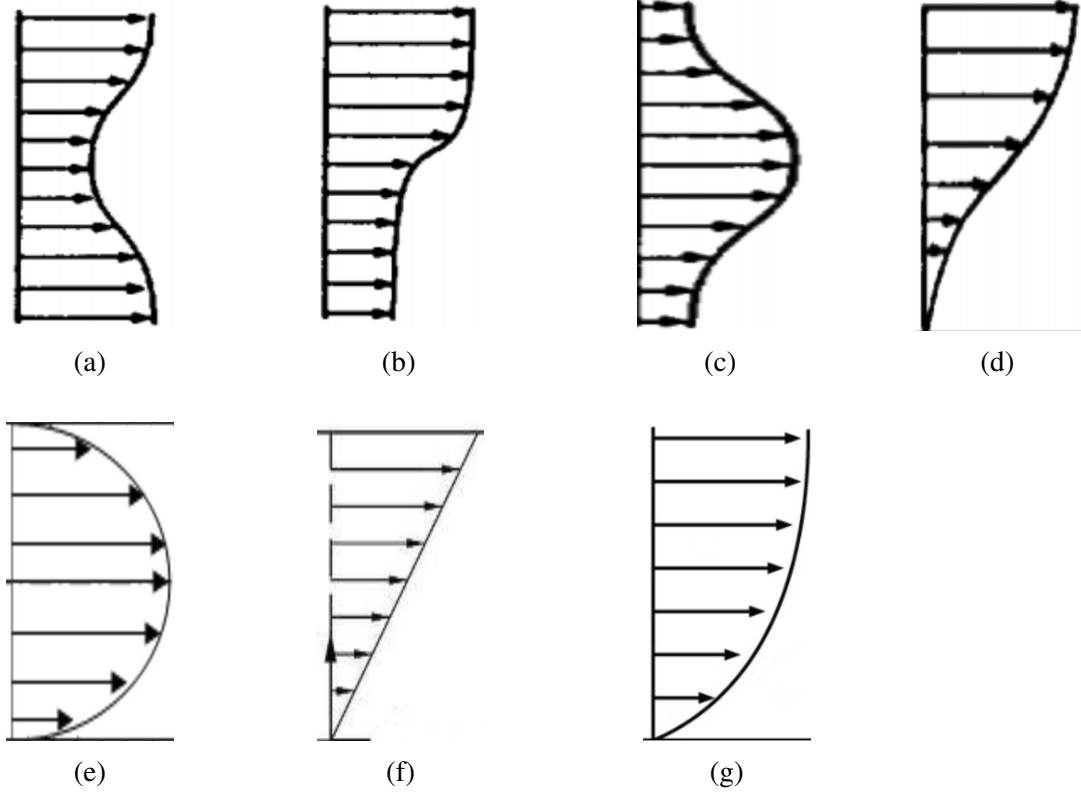


Figure 2.1: The velocity distributions (a) wake, (b) mixing layer, (c) jet and (d) Blasius profile with an adverse pressure gradient are inviscidly unstable, whereas, (e) Poiseuille flow, (f) Couette flow and (g) Blasius flow with favorable pressure gradient are unstable only in the viscous limit.

point criterion. The spanwise mean vorticity profile ($D\bar{U}$) corresponding to the mean velocity distribution in Fig. 2.2a has a local minimum and hence Fjortoft's criterion is not satisfied.

In general, neither the Rayleigh's criterion nor the Fjortoft's theorem is sufficient for inviscid instability, for instance, a flow bound by two walls which are in relative motion. To prove this point, Lin [28] considered a sinusoidal velocity distribution $\bar{U}(y) = A + B \sin y$; $y_1 < y < y_2$, which has an inflection point at $y = 0$. He showed that in spite of the existence of the inflection point, as long as the width of the domain is restricted to $y_2 - y_1 < \pi$ the flow will always be stable. However, for velocity distributions of symmetrical or boundary layer type, Tollmien [27] proved that the inflection point criterion is sufficient for the appearance of unstable or neutral disturbances. A neutral disturbance can exist when the phase speed of the perturbation is equal to the mean flow velocity corresponding to the inflection point. In addition, self-excited disturbances appear when the wavenumber of the perturbation is less than the wavenumber (α_I) corresponding to the phase speed at the inflection point. Disturbances are damped when $\alpha > \alpha_I$.

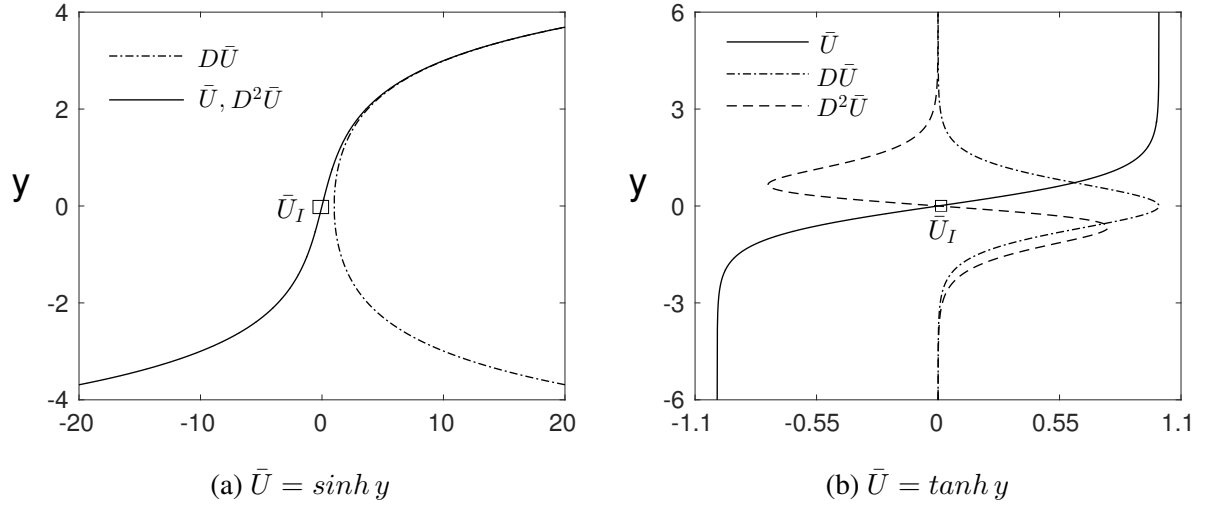


Figure 2.2: Two different velocity distributions along with their corresponding first and second derivatives with respect to y are plotted. Both the profiles have a single inflection point at $y = 0$, marked by the rectangle. (a) $(\bar{U} - \bar{U}_I) D^2 \bar{U} \geq 0$, therefore stable according to Reyleigh-Fjortoft's criterion; (b) possibly unstable since $(\bar{U} - \bar{U}_I) D^2 \bar{U} \leq 0$.

2.2 Inviscid instability: Compressible flow

Lees and Lin [41] were the first ones to formulate the linear stability equations for the laminar compressible boundary layer, followed by Dunn and Lin [42], who proposed a simplified form of these equations. An excellent review of the instability of compressible flows can be found in Mack [43]. Subsequently, Mack formulates the inviscid theory and performs computations to investigate the inviscid instability of high-speed shear flows [44].

It is shown that a compressible boundary layer flow is unstable at both viscous and inviscid limit. The first mode, also known as the vorticity mode, is the same Tollmien-Schlichting (TS) mode that is observed in low-speed flows. When the Mach number increases, the viscous instability mechanism responsible for the TS waves vanishes, so the first mode behaves as an inviscid mode. It arises due to a discontinuity in Reynolds stress across the critical layer. The growth of the TS mode is small and takes place over a viscous length scale in a two-dimensional boundary layer. As reported by Lees and Lin [41], the appearance of a generalized inflection point in the mean velocity profile is a sufficient criterion for instability of the first mode in a compressible flow. The authors proved it by studying the integrated value of internal and kinetic energy in the boundary layer. In a compressible flow, the angular momentum of a given volume of the fluid is a function of the product of vorticity and mean flow density. Therefore, the gradient of this quantity $(\frac{\partial}{\partial y}(\bar{\rho} \frac{\partial \bar{U}}{\partial y}))$ has the similar role as the curvature of the mean velocity distribution in the case of incompressible flows. The generalized inflection point is located

where,

$$\frac{\partial}{\partial y}(\bar{\rho} \frac{\partial \bar{U}}{\partial y}) = 0. \quad (2.7)$$

A compressible, insulated flat plate boundary layer has a generalized point of inflection and hence not stable in the inviscid limit. As the surface is cooled, another inflection point appears in the flow. At a fixed Mach number, a certain level of cooling can move the inflection points closer until they cancel one another. After this, the unstable first mode disturbances will not exist. At the generalized point of inflection, the wave speed of the inviscid neutral mode can be equal to the streamwise component of the boundary layer velocity, and it is adjacent to unstable modes in wavenumber space.

2.2.1 Mack's acoustic modes

In a high-speed flow, disturbances can be sonic, supersonic or subsonic depending on whether the phase speed of the disturbance in comparison to the free-stream velocity is equal to, higher or less than the speed of sound, calculated based on the mean flow quantities. Mack [45] used a complex quantity called the relative Mach number to quantify the behavior of the disturbances in terms of the wavenumber and mean flow quantities. The component of the mean flow velocity ($\vec{V}^*$) projected along the wavenumber vector ($\vec{\kappa}^*$) is,

$$\vec{V}^* \cdot \frac{\vec{\kappa}^*}{|\vec{\kappa}^*|} = (\bar{U}^* \hat{i} + \bar{W}^* \hat{j}) \cdot \frac{(\alpha^* \hat{i} + \beta^* \hat{j})}{|\vec{\kappa}^*|} = \frac{\bar{U}^* \alpha^* + \bar{W}^* \beta^*}{|\vec{\kappa}^*|},$$

here, $\bar{U}^*$ and $\bar{W}^*$ represent the velocity components in the stream wise (x) and spanwise (z) directions respectively. The wavenumber components along the x and z directions are represented as α^* and β^* respectively, and $\hat{i}$ and $\hat{j}$ are the corresponding unit vectors. Now, the Mach number relative to the complex phase speed c^* and along the wavenumber vector can be expressed as,

$$M_r^* = \frac{\bar{U}^* \alpha^* + \bar{W}^* \beta^* - c^*}{\sqrt{\gamma R T^*}}.$$

All the quantities used in the definition of the relative Mach number in the above expression are dimensional. When the velocity components and the phase speed of the disturbance are non-dimensionalized by the free-stream velocity, and the mean temperature with respect to its free-stream value, the expression for the relative Mach number becomes,

$$M_r = M_\infty \frac{\bar{U}\alpha + \bar{W}\beta - \omega}{|\vec{\kappa}| \sqrt{\bar{T}}}, \quad (2.8)$$

here, M_∞ and ω denote the free-stream Mach number and disturbance frequency respectively. Now, if we consider only the real parts of disturbance frequency and wavenumber, Eq. 2.8 becomes,

$$\begin{aligned} M_c &= M_\infty \frac{\bar{U}\alpha_r + \bar{W}\beta_r - \omega_r}{|\vec{\kappa}_r| \sqrt{\bar{T}}} \\ &= M_\infty \frac{\bar{U}\cos\theta + \bar{W}\sin\theta - c_r}{\sqrt{\bar{T}}}, \end{aligned}$$

and M_c is defined as the convective Mach number of the flow. The subscript r is used to denote the real part of the quantities. In the above expression, it is assumed that the perturbation is aligned at an angle θ to the free-stream. The wavenumber components α_r and β_r can be written as $|\vec{\kappa}_r| \cos\theta$ and $|\vec{\kappa}_r| \sin\theta$ respectively, and the phase speed is defined as,

$$c_r = \frac{\omega_r}{\sqrt{(\alpha_r^2 + \beta_r^2)}} = \frac{\omega_r}{|\vec{\kappa}_r|}.$$

Now, the instability waves are termed as supersonic when $|M_c| > 1$, which implies that there are regions in the flow where $M_c < -1$ or $M_c > 1$ or both. In a low-speed flow, the phase velocity of the disturbance is uniquely related to the wavenumber. However, in the case of high-speed flows, when the mean flow velocity is supersonic with respect to the disturbance phase speed, there can be a series of wavenumbers having the same phase speed ([45], [50]-[53]). These additional modes are known as the higher modes or Mack modes and the inviscid theory states that the most unstable mode in a boundary layer is the second Mack mode (first of the higher modes). This makes these modes more relevant to the boundary layer stability. Since there are no locally supersonic disturbances in an incompressible flow, therefore it does not have any instability modes analogous to Mack modes. The reason for the additional modes can be seen easily from an examination of the second-order inviscid stability equation (compressible Rayleigh equation) for the pressure eigenfunction ($\hat{p}$), which is,

$$D^2\hat{p} - \left(\frac{2(\alpha D\bar{U} - \beta D\bar{W})}{\alpha \bar{U} + \beta \bar{W} - \omega} - \frac{D\bar{T}}{\bar{T}} \right) D\hat{p} - \kappa^2(1 - M_c^2)\hat{p} = 0. \quad (2.9)$$

The above equation is obtained by assuming a parallel flow and also heat conduction and viscous terms for the disturbances are neglected. The inviscid theory is used here since in a high-speed flow, with an increase in the Mach number, the amplification rate of the disturbance increases with Reynolds number. Therefore, to understand the stability characteristics of a compressible boundary layer flow at high Mach numbers, the inviscid theory is much more appropriate.

Mack was the first to discover that the mathematical nature of Eq. 2.9 changes from elliptic to hyperbolic when there is a domain of relative supersonic flow. The wave nature of Eq. 2.9 at high Mach numbers gives rise to multiple discrete eigenvalues, which are known as Mack modes. Since the region of relative supersonic flow is maximum when these modes are of two-dimensional nature, therefore Mack modes are more unstable when they are two-dimensional. These modes have much higher frequencies than the first mode or the Tollmien-Schlichting mode. Mack modes can be characterized from their behavior near the wall in a boundary layer. The presence of sound wave reflections between the relative sonic line ($M_c = \pm 1$) and the wall confirms the existence of Mack modes in a flow. Lees coined these modes as acoustic modes since they lead to the acoustic trapping of energy near the wall. The second mode which is the most unstable amongst the Mack modes is detected and measured in a lot of wind tunnel experiments ([46] - [48]). Guschin & Fedorov [49] mentioned that when the fast and the slow modes of the discrete spectrum synchronize with each other then the second mode instability appears.

Fig. 2.3a shows the contour plot of pressure perturbation corresponding to the second mode instability in a flat plate flow at Mach 6. To identify the region of relative supersonic flow, we plot the variation of the convective Mach number across the boundary layer in Fig. 2.3b. It can be seen that relative sonic line ($M_c = -1$) confines the disturbances within the relative supersonic region and hence it can be treated as an acoustic waveguide (see Fig. 2.3a). Above the relative sonic line, the phase speed of the disturbance is subsonic relative to the mean flow velocity. However, below the sonic line, the disturbances travel at a supersonic speed with respect to the mean flow (see Fig. 2.3b). Due to this, sound waves are of oscillatory nature inside the relative supersonic region and transition to evanescent waves above the sonic line. This causes a part of the wave to reflect back inside the boundary layer and a change in phase is observed for the perturbations across the relative sonic line (see Fig. 2.3a). In the case of higher Mack modes, the number of oscillations increase with an increase in the mode number and an

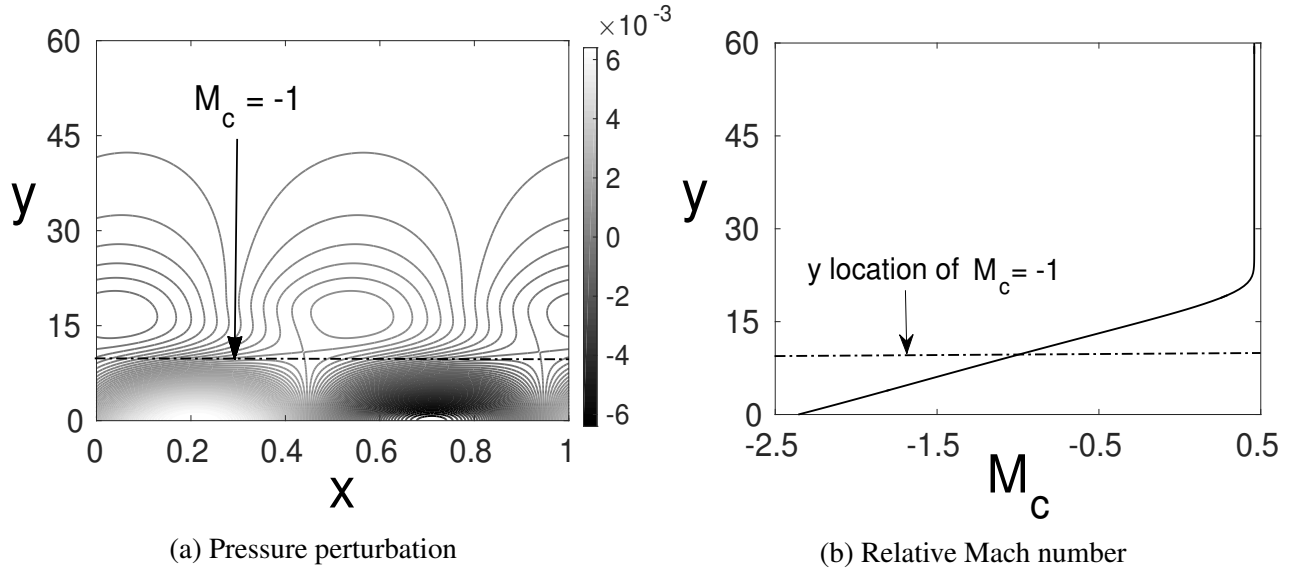
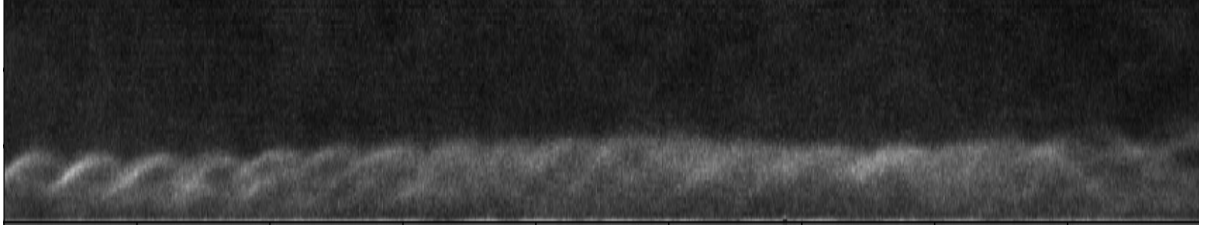


Figure 2.3: (a) The contour plot of pressure eigenfunction for second mode instability in the boundary layer over a flat plate at Mach 6, and Reynolds number and Prandtl number of 2.5×10^7 and 0.4 respectively. (b) The variation of the relative Mach number corresponding the second mode is shown to identify the region of relative supersonic flow in the boundary layer (the dashed line separates the supersonic region (lower section) of flow from the subsonic one).

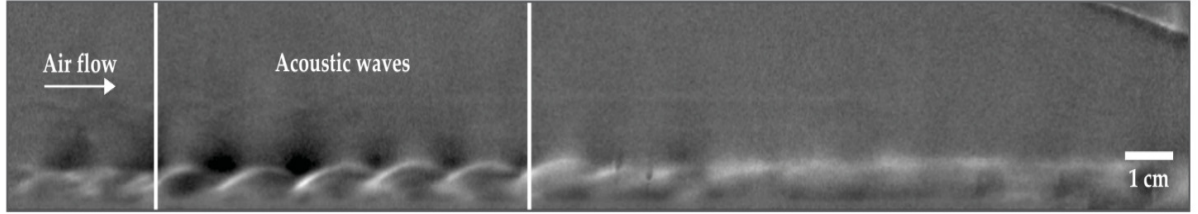
additional zero appears in the pressure eigenfunction [60].

Acoustic modes are observed in many transition experiments in high-speed flows, and their role in laminar to turbulent transition has been widely acknowledged. In a similar experiment, Casper et al. [54] observed that the boundary layer over a 7° half-angle cone is fully dominated by second mode disturbances at Mach and Reynolds number of 8 and $7.1 \times 10^6/m$ respectively (see Fig. 2.4a). These instability waves have grown significantly and filled the entire Schlieren viewing area. Ref. [55] also reports similar waves trapped inside the boundary layer of a circular cone at a Mach number of 10 (see Fig. 2.4b). These instability waves noticed in both the cones are same as that we observed in the boundary layer over a flat plate at $M = 6$ in Fig. 2.3a. The alternating dark and light patches are seen in Fig. 2.3a and 2.4 represent the high and low amplitude of pressure eigenfunction. The acoustic modes grow very rapidly and lose their periodicity in the downstream region of the cones, indicating the transition to turbulence.

The acoustic modes are important since the flow is unstable to these two-dimensional inviscid disturbances in spite of the presence of inflectional disturbances. Further, compared to the vorticity mode, the amplification rate of the acoustic modes are significantly higher at high Mach numbers. These modes are observed in supersonic free-shear layers ([53], [56], [57]) such as jets, wakes, supersonic mixing layers in addition to the boundary layers and bounded



(a) Taken from [54]



(b) Taken from [55]

Figure 2.4: (a) Schlieren image showing the Mack's second mode in a boundary layer over a 7° half-angle cone at a Mach number of 8 and Reynolds number of $7.1 \times 10^6/m$; (b) Schlieren image of acoustic waves in a Mach 10 boundary layer along the surface of a 1.55 m long circular cone.

flows (Couette, Channel flow, etc.). Mack reported that viscosity always stabilizes the Mack modes and the maximum growth rate of these modes is observed at an infinite Reynolds number.

2.3 Effect of wall cooling

It is proven numerically [44] and experimentally [58] that wall cooling can be used to completely stabilize the vorticity or the first mode, by removing the inflectional points from the boundary layer profile. However, cooling increases the amplification of the acoustic modes, especially the second mode, and this destabilization can be attributed to the decrease in the boundary layer thickness as the surface of the vehicle is cooled. This can also be seen from Fig. 2.5, which shows the effect of cooling the wall on the spatial growth rate of the first two modes at varying Mach numbers. The most unstable second mode disturbance has a wavelength which is approximately two times the thickness of the boundary layer. Both viscosity of the fluid and wall-cooling influence the boundary layer thickness. As the boundary layer becomes thinner, there appear more sound waves per unit distance, and since the growth rate per unit wavelength remains almost the same, the amplification rate of the acoustic mode increases [60]. Mack [44] also mentioned that the behavior of the second mode is strongly affected by the size of the re-

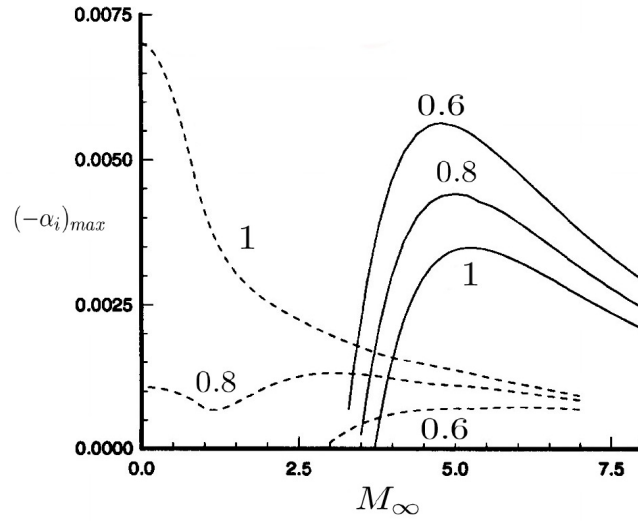


Figure 2.5: Variation of the maximum growth rate of instability waves as a function of the Mach number for a flow over a flat plate at three wall temperature ratios (T_w/T_{ad}). Reynolds number $\sqrt{Re} = 1500$, and a Prandtl number of 0.72 are considered. The dashed and solid lines represent the first mode and the second mode respectively. The figure is taken from [59].

gion of relative supersonic flow. Therefore, while examining the second mode in a high-Mach number flow, we should focus on the factors which can influence the relative supersonic region and the thermal boundary layer profile.

2.4 Viscous instability

The disturbances are called viscous instability when their amplification rate increases with a decrease in the Reynolds number. Initially, it was believed that viscosity only stabilizes the flow, but in 1921 Prandtl [61] showed that viscosity can also be destabilizing. Instabilities appear only when viscosity acts to generate a positive value of Reynolds stress close to the wall. The viscous theory is well documented by Mack in [43] and his earlier comprehensive article in [62]. For two-dimensional compressible boundary layers, it was proved that viscous instability decreases with Mach number ([52], [63]).

Since the Blasius boundary layer does not possess an inflection point, only the action of viscosity can make it unstable. To examine the effect of viscosity, we can derive an evolution equation for the kinetic energy of the disturbance (Reynolds-Orr Equation), which is,

$$\frac{d}{dt} \int_V \frac{\tilde{u}_i \tilde{u}_i}{2} dV = - \int_V \tilde{u}_i \tilde{u}_j \frac{\partial \bar{U}_i}{\partial x_j} dV - \frac{1}{Re} \int_V \frac{\partial \tilde{u}_i}{\partial x_j} \frac{\partial \tilde{u}_i}{\partial x_j} dV. \quad (2.10)$$

The term on the left of the above equation denotes the rate of change of disturbance kinetic energy. The production of disturbance energy appears on the right-hand side of Eq. 2.10 (the first term) and it indicates how much energy is transferred to the disturbances $\tilde{u}_i$ from the mean flow $\bar{U}_i$ through the interaction of Reynolds stress $\tilde{u}_i\tilde{u}_j$ and the mean flow shear $\frac{\partial \bar{U}_i}{\partial x_j}$. The last term stands for the energy dissipation due to viscous effects. It should be noted that if the velocity components $\tilde{u}_i$ and $\tilde{u}_j$ are not in phase, the average value of Reynolds stress over a period is zero. To induce instability, it is essential that the stress must have the same sign as that of mean shear. Viscosity changes the phase relationship between the velocity components $\tilde{u}_i$ and $\tilde{u}_j$, leading to a positive value of production, and when it is larger than the viscous dissipation, the flow becomes unstable.

2.5 Summary

In this chapter, we reviewed various inviscid and viscous mechanisms leading to flow transition in the compressible and incompressible limit. It is shown that inflectional velocity profiles satisfying Fjortoft's criterion can be unstable in the inviscid limit. However, mean flow profiles without an inflection point can also be unstable due to the effect of viscosity, for example, the Blasius boundary layer. Viscosity is responsible for changing the phase relationship between the velocity components of the disturbance, leading to instability. In the case of incompressible flows, only one instability mode, T-S waves, which is viscous in nature is associated with flow transition. In addition to the first mode or the vorticity mode, in a compressible flow there can have a large number of modes if there is a region of relative supersonic flow. These are known as Mack or acoustic modes and are inviscid in nature and appear due to the trapping of sound waves inside the boundary layer. Cooling can be used to completely stabilize the first mode, however, it is found to destabilize the higher modes.

Chapter 3

Methodology

3.1 Couette flow

The flow considered in this work is a perfect Newtonian gas, confined between two parallel plates, with the top plate moving with a velocity of U_∞ and the lower wall is at rest. The distance between the two plates and the top wall quantities are used to non-dimensionalize the flow variables. The assumption of steady and uni-directional flow simplifies the governing equations for the Couette flow and they can be written as,

x momentum:

$$\frac{\partial}{\partial y}(\bar{\mu} \frac{\partial \bar{U}}{\partial y}) = 0, \quad (3.1)$$

energy equation:

$$\frac{\partial}{\partial y}(\bar{\kappa} \frac{\partial \bar{T}}{\partial y}) + (\gamma - 1) M^2 Pr \bar{\mu} (\frac{\partial \bar{U}}{\partial y})^2 = 0. \quad (3.2)$$

Here $\bar{U}$, $\bar{T}$, $\bar{\mu}$, $\bar{\kappa}$ and γ represent streamwise velocity, temperature, viscosity, thermal conductivity and the ratio of specific heats respectively. The wall-normal coordinate is denoted by y . Pressure has been non-dimensionalized with respect to its top wall value. The reference Prandtl number, Mach number, and the Reynolds number are defined in terms of the top wall parameters and are represented by Pr , M and Re respectively. The overbar is used to indicate the mean flow quantities. Velocity satisfies the no-slip condition at both the walls. The temperature boundary conditions imposed on the upper and lower walls are isothermal and adiabatic respectively.

To examine the physical effects of viscosity and thermal conductivity, we have considered

three different model problems. These are – stratified viscosity flow, stratified conductivity flow and stratified viscosity, and conductivity flow, and we denote them as μ -stratified, κ -stratified and μ - κ -stratified flow, respectively. In a μ -stratified flow, only the variation of viscosity is allowed, and conductivity is kept at its reference value. In a κ -stratified flow, the conductivity varies as a function of temperature, and viscosity is maintained at its reference value. A μ - κ -stratified flow allows the variation of viscosity and thermal conductivity simultaneously, as a function of temperature. We use Sutherland's law to compute the viscosity and conductivity at varying temperature. It is consistent with the current framework of a compressible perfect gas flow, without any high-temperature effects such as dissociation and ionization. We also define another flow with constant values (= top wall value) of viscosity and conductivity to determine the effect of a change in the mean values of these two transport properties, which is referred to as the reference flow. A more detailed explanation of these flows is available in the subsequent chapters.

3.2 Details of stability analysis

3.2.1 Linear stability theory

Instantaneous flow quantities (A), such as velocity, density or temperature, are expressed as a sum of the mean flow variable ($\bar{A}$) and a small fluctuation quantity ($\tilde{A}$), i.e.,

$$A = \bar{A}(y) + \tilde{A}(x, y, z, t). \quad (3.3)$$

The linearized equations for the disturbances are obtained by substituting Eq. 3.3 into the non-dimensional form of the governing equation and neglecting the non-linear and higher order terms in the fluctuations (see Appendix I). We assume a normal mode form for the fluctuations as,

$$\tilde{A}(x, y, z, t) = \hat{A}(y) e^{i(\alpha x + \beta z - \omega t)}, \quad (3.4)$$

where $\hat{A}$, α , β and ω represent the amplitude, streamwise and spanwise wavenumber and frequency of the disturbance respectively. A temporal stability analysis is carried out in this work, which requires ω to be complex and α , β both are real parameters. Substituting Eq. 3.4 into the disturbance equations, we obtain a homogeneous system of ordinary differential equations,

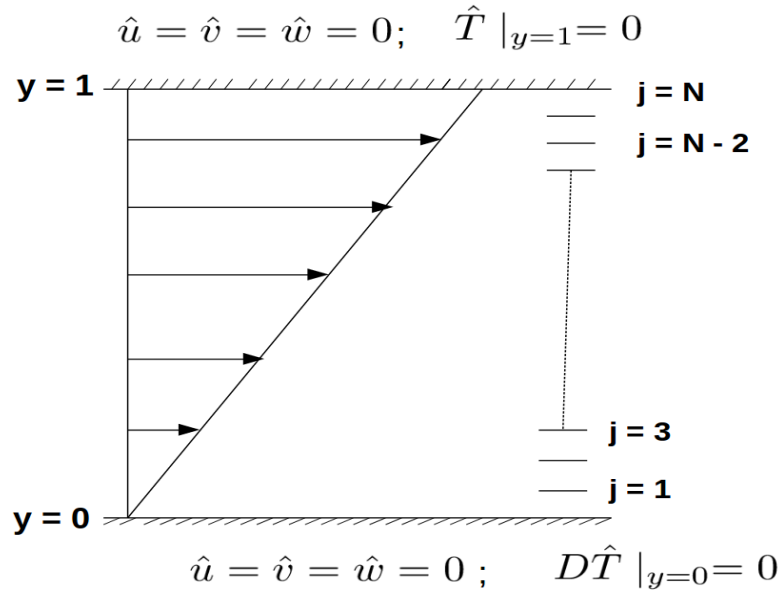


Figure 3.1: Schematic of a plane Couette flow showing the grid point distribution, along with the boundary conditions on the disturbance perturbation at the wall.

which can be expressed as,

$$B\psi = \omega I \psi. \quad (3.5)$$

where $\psi = (\hat{u}, \hat{v}, \hat{w}, \hat{\rho}, \hat{T})^{tr}$ represents the complex amplitude of the disturbance, and I is the identity matrix. The elements of matrix B are mentioned in the appendix 1, and they are functions of α, β, Re, Pr, M and the mean-flow variables.

The velocity fluctuations satisfy the no-slip boundary condition at both the walls. Isothermal and adiabatic boundary conditions are applied to the temperature fluctuations at the upper and the lower walls respectively (see Fig. 3.1). Since it is not possible to impose a physical boundary condition on density fluctuations, we have to procure its value by using the state equation such that the boundary condition on pressure is satisfied at the wall. The boundary condition on pressure is obtained by solving the momentum equation for wall-normal velocity, which leads to a Neumann boundary condition for pressure ($D\hat{p}$, where D represents the first order derivative with respect to y). The normal derivative of pressure ($D\hat{p}$) is expressed in terms of velocity and temperature perturbations, which are known at the wall. Computationally this is achieved by replacing the continuity equation at both the boundaries ($y = 0$ and $y = 1$) with the y -momentum equation in the matrix A and B (see Appendix I).

Linear stability analysis is performed by transforming the set of differential equations to linear algebraic equations using a spectral representation. The eigenvalues are obtained by solving the characteristic determinant of a generalized eigenvalue problem using the global

method. The QZ algorithm of MATLAB provides the complex frequency $\omega = \omega_r + i\omega_i$, where ω_i represents the growth or decay rate of the disturbance. Compared to the local methods, global methods are more expensive, since all the eigenvalues are calculated simultaneously. Also, it is important to attain spectral convergence and great care must be exercised to preclude any spurious eigenvalue. We have used around 150 – 200 Chebyshev points to compute the eigenvalues for Couette flow, and the results presented in this report are grid converged (see Appendix III).

We note that the linear stability results presented in the report are for $\beta = 0$. This is based on the finding that two-dimensional disturbances are the least stable modes for compressible Couette flows, as reported by Malik et al. [22] for systematically varying β values (see Fig. 7 in Ref. [22]).

Numerical method

Chebyshev spectral method is suitable for the numerical approximation of non-periodic boundary value problems as it has low phase errors and high precision. In this method, numerical solutions satisfy the governing equations at collocation points. The interpolation points are known as the Chebyshev-Gauss Lobatto points and are calculated as, $x_j = \cos(\frac{j\pi}{N})$ and $j = 0, 1, \dots, N$. Any function $u(x, t)$ on the domain $[-1, 1]$ can be expressed as,

$$u_N(x, t) = \sum_{k=0}^N u_k^\dagger(t) T_k(x), \quad (3.6)$$

here T_k represents the Chebyshev polynomial of order N , and can be expressed as $T_N(x) = \cos(N \cos^{-1}(x))$. $u_k^\dagger$ is known as the expansion coefficient and is evaluated as,

$$u_k^\dagger(t) = \frac{2}{Nc_k} \sum_{j=0}^N \frac{u_N(x_j, t)}{c_j} \cos\left(\frac{j\pi k}{N}\right), \quad k = 0, 1, \dots, N. \quad (3.7)$$

Here, $c_0 = c_N = 2$ and 1 otherwise. At the collocation points (x_j) , the spatial derivatives of $u_N(x, t)$ can be computed as,

$$u'_N(x_j, t) = \sum_{k=0}^N D_{jk} u(x_k, t), \quad j = 0, 1, \dots, N, \quad (3.8)$$

here D is the Chebyshev collocation derivative matrix and is given by:

$$D_{jk} = \begin{cases} \frac{2N^2 + 1}{6}, & j = k = 0 \\ -\frac{2N^2 + 1}{6}, & j = k = N \\ -\frac{x_k}{2(1 - x_k^2)}, & j = k = 1, \dots, N - 1 \\ \frac{c_j(-1)^{j+k}}{c_k(x_j - x_k)}, & j \neq k \end{cases}$$

3.2.2 Transient growth analysis

Non-modal analysis or transient growth is used to study the short-term dynamics of a system. The parameter space where the flow is predicted to be linearly stable is the region of interest for transient growth. It investigates the possibility of transient amplification of initial disturbance energy, in a flow that supports only decaying eigenvalues and is thus linearly stable. Cases, where such transient amplification can trigger non-linearities and even cause a linearly stable flow to undergo a transition to turbulence are now well known, e.g., incompressible Couette flow. In general, most of the flows are modally stable at low Reynolds numbers, and therefore non-modal growth is the only possible path to cause flow transition. However, at higher Reynolds numbers, a competition between modal and non-modal growth mechanisms determines the route to flow transition. If the short-time dynamics of transient growth are able to surpass the exponential amplification of unstable modes, then the non-modal growth will lead to flow turbulence. Although modal instabilities don't exist for low Mach numbers and highly cooled walls (the first mode is highly damped due to wall-cooling, and the second mode appears only at high Mach number), transition to turbulence can still be initiated by infinitesimal perturbations because of the large density gradients introduced by wall-cooling, which can result in significant transient energy amplification. However, as the Mach number is increased to around 5 (which may be observed in a shock tunnel operating at a moderate enthalpy), the second mode instability dominates, leading to a higher value of modal N factor compared to the non-modal case. Increasing the Mach number decreases the level of transient energy amplification, and hence the modal N factor overtakes the non-modal N factor at a relatively low Reynolds number. Experiments predict a transition Reynolds number of around 2-3 million for a flat plate boundary layer, and at this Reynolds number, disturbance growth predicted by the linear stability theory has an order of magnitude higher amplification compared to the transient

growth calculation. Therefore, at high Mach numbers along with a cold wall boundary condition, relevant to high enthalpy flows, transition is most likely to occur through an exponential growth of instabilities alone.

In a compressible medium, the velocity field is coupled with temperature, therefore, the changes in temperature and density have to be considered. A suitable norm for a compressible flow is the one which accounts for internal energy changes in addition to the disturbance kinetic energy. The norm has to be such that it can eliminate energy transfer associated with pressure since the compression work is conservative [64]. Therefore, for evaluating the temporal energy growth, we use Mack's energy norm, which is

$$2E = \int_0^1 \bar{\rho} (|u_s|^2 + |v_s|^2 + |w_s|^2) + \frac{\bar{T}}{\gamma \bar{\rho} M^2} |\rho_s|^2 + \frac{\bar{\rho}}{\gamma(\gamma - 1) \bar{T} M^2} |T_s|^2 dy, \quad (3.9)$$

here the subscript s is used to denote the state vector ($\phi = [u_s, v_s, w_s, \rho_s, T_s]^{tr}$), which can be expressed in terms of the eigenvectors (ψ) and the expansion coefficients (χ) as,

$$\phi = \sum_{k=1}^n \chi_k(0) \exp(-i\omega_k t) \psi_k, \quad (3.10)$$

and n is the number of eigenvalues used to compute the transient energy growth. The series should be truncated such that the transient growth results do not vary as we include additional eigen modes. In terms of eigenvector expansion, Eq. 3.9 can be written as,

$$2E = (\Lambda \chi)^H \left[\int_0^1 Q^H \mathbf{M} Q dy \right] \Lambda \chi. \quad (3.11)$$

Here, H denotes the Hermitian matrix, and Λ is a diagonal matrix with elements $e^{-i\omega t}$. The columns of matrix Q contains the eigenvectors in the truncated vector space, and the matrix $\mathbf{M}$ includes the coefficients of the disturbances in Eq. 3.9. Due to its positive definite nature, we can write the integral in Eq. 3.11 as a product of matrix F and its Hermitian F^H . Using the eigenvector basis, F can be calculated using the Cholesky decomposition. The energy norm can be simplified in terms of matrix F and the eigenvector expansion coefficients as,

$$2E = \|F \Lambda \chi\|^2. \quad (3.12)$$

Now, the optimal transient growth of disturbance energy is given by,

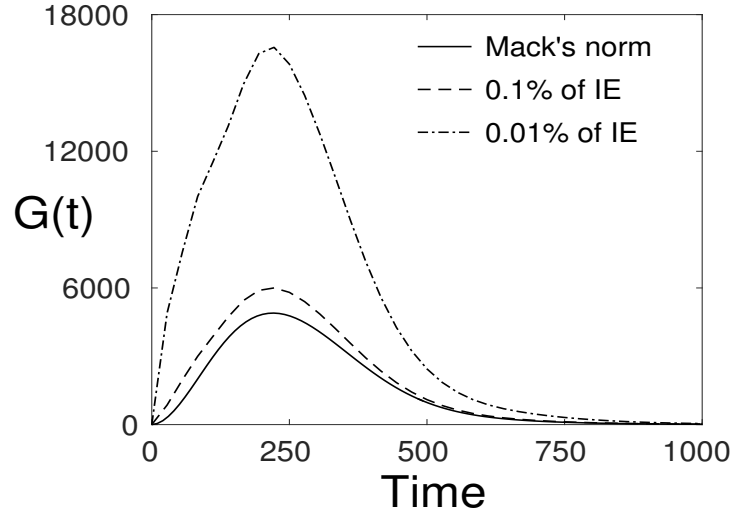


Figure 3.2: Comparison of the energy amplification factor, $G(t)$ obtained with three different norms for a viscosity and conductivity stratified flow at $M = 10$, $Pr = 0.2$, $Re = 10^5$, $\alpha = 0.05$ and $\beta = 3.5$.

$$G = \max \frac{E(t)}{E(0)} = \max \frac{\|F \Lambda \chi\|^2}{\|F \chi\|^2} = \|F \Lambda F^{-1}\|^2, \quad (3.13)$$

which is determined by the singular value decomposition. The right singular vector corresponding to the largest singular value provides the expansion coefficients of the optimal perturbation. For a certain combination of streamwise and spanwise wavenumber, we can obtain the maximum energy amplification at a given time $G(\alpha, \beta, t)$. The perturbation energy can also be optimized over a specific period of time $G_{max}(\alpha, \beta)$. More detailed discussions on transient growth can be found in Refs. [64] - [67]. We have used the eigenvalues with a decay rate greater than 0.5 ($\omega_i > -0.5$) to compute the transient energy amplification for all the cases presented in this report.

The choice of the norm can make a significant difference to the transient growth results. If the norm consists of only the kinetic energy term, without the contributions of thermodynamics perturbations, it can lead to much higher transient energy amplification compared to that obtained using Mack's norm. To show this, we have plotted the variation of energy amplification factor with time for a viscosity and conductivity stratified Couette flow in Fig. 3.2. The parameters considered in the figure correspond to the highest transient energy gain in Fig. 7.4f. The solid line in Fig. 3.2 is computed using Mack's energy norm proposed for compressible flows (Eq. 3.9). The remaining two lines are obtained by considering 0.1% and 0.01% of the total internal energy of Mack's energy norm. Although the variation of $G(t)$ for the three cases

are qualitatively similar, there is a significant variation in the peak transient energy amplification. The peak value of $G(t)$ increases around 3.5 times when we consider only 0.01% of the energy associated with the thermodynamic fluctuations (density and temperature). The results are akin to those presented by Tempelmann et al. [68], which reports large differences in the optimal growth factor computed using kinetic energy norm and Mack's norm in a compressible boundary layer flow. The authors also highlight the importance of including the thermodynamic fluctuations in the calculation of total energy of the disturbance.

Summary

The governing equations for a compressible plane Couette flow in non-dimensional form are derived from the three-dimensional Navier-Stokes equations. To isolate and study the effect of transport properties on flow stability and transition, four different model problems are constructed. In the first case, viscosity and conductivity are maintained at their corresponding top wall values, and this flow is referred to as the reference flow. In the next model problem, the viscosity is kept at its top wall value, and the variation of thermal conductivity is allowed, whereas in the third model problem, viscosity varies as a function of temperature, and conductivity is kept at its reference value. The second case will provide only the effect of variation in conductivity, and the third case examines the effect of variation in viscosity. We have formulated one more case where both the transport properties are allowed to vary independently, as a function of temperature, and it brings out the effect of coupled variations in viscosity and thermal conductivity. A detailed description of different stability approaches used to perform the stability analysis of the base flow is also discussed. The formulation of the eigenvalue problem along with the method used to discretize the mean flow as well as the perturbation equations is also reviewed.

Chapter 4

Constant property flow

In this chapter, we discuss the governing equations for a Couette flow with uniform viscosity and thermal conductivity. A detailed stability analysis is carried out for this constant shear flow, using both linear stability theory and transient growth.

4.1 Base flow

For obtaining the governing equations for a plane Couette flow with uniform viscosity ($\bar{\mu} = 1$) and thermal conductivity ($\bar{\kappa} = 1$), flow is assumed to be steady and unidirectional. This leads to identically zero solutions for the continuity and z-momentum equations, and the y-momentum equation provides a constant pressure condition ($\bar{p} = 1$). The x-momentum (Eq. 3.1) and energy equations (Eq. 3.2) simplify to:

$$\frac{d\bar{U}}{dy} = \text{constant}, \quad (4.1)$$

$$\frac{d}{dy} \left(\frac{d\bar{T}}{dy} \right) + (\gamma - 1) M^2 Pr \left(\frac{d\bar{U}}{dy} \right)^2 = 0. \quad (4.2)$$

Application of non-slip boundary conditions at the wall simplifies Eq. 4.1 to $\bar{U} = y$ and is plotted in Fig. 4.1a for a $M = 6$ flow. Using this profile for velocity Eq. 4.2 can be integrated to obtain the following expression for temperature,

$$\bar{T} = 1 + \frac{(\gamma - 1) M^2 Pr}{2} (1 - y^2). \quad (4.3)$$

The temperature variations for two Mach numbers (6 and 12) corresponding to a reference Prandtl number of 0.5 are shown in Fig. 4.1b. We observe that starting from the reference value

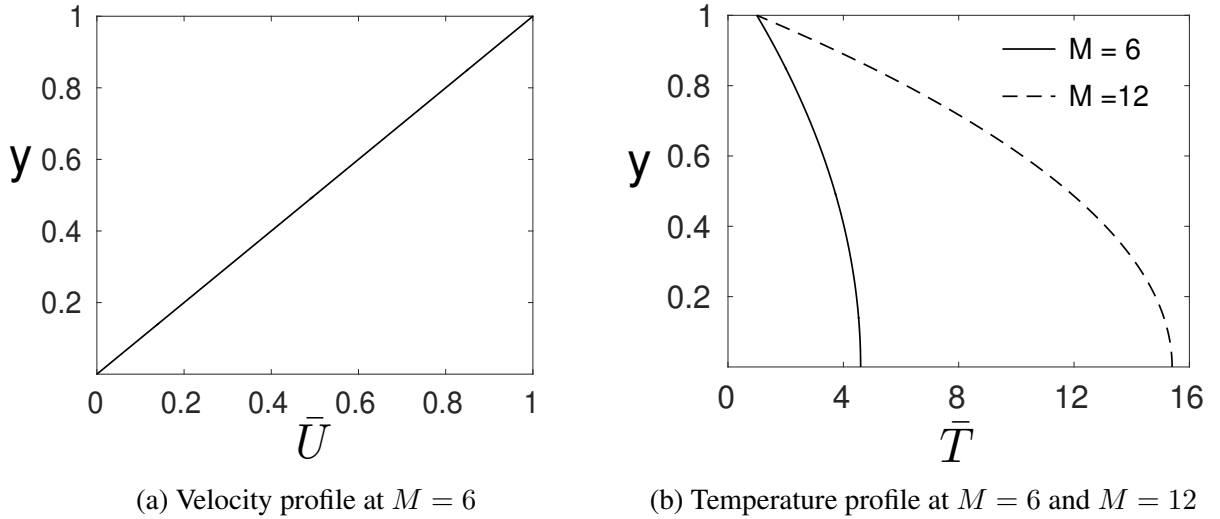


Figure 4.1: Mean flow profiles along the wall-normal direction for a plane Couette flow with uniform viscosity and conductivity at a reference Prandtl number of 0.5.

(= 1) at the top wall, temperature monotonically increases and attains a maximum value at the bottom wall, satisfying the adiabatic wall condition. It can also be noted that increasing the Mach number leads to an enhancement in the temperature value.

A slight departure in the computation of the mean flow profiles can lead to a large variation in the stability results. Therefore the basic state along with its spatial derivatives should be calculated with high precision. Sufficient grid points are required near the wall (around 50 Chebyshev points), in addition to the critical layer and around the inflection point. With an increase in the Mach number, the critical layer moves towards the boundary layer edge, therefore special care has to be taken to have a proper clustering of collocation points. The use of the proper boundary condition is also very important since it only defines the problem.

4.2 Linear stability results

The parameters which govern the stability characteristics of compressible Couette flow are Mach number, Reynolds number, Prandtl number and the wavenumber of the disturbance. The effect of M , Re and wave number on flow stability have already been studied in [20]. Here, we quantify the effect of Prandtl number on the eigenspectrum.

Fig. 4.2 plots the real (c_r) and imaginary (c_i) parts of the temporal eigenvalues for a Mach 12, $\alpha = 5$, $Re = 10^5$, plane Couette flow with two-dimensional disturbances ($\beta = 0$). The set of eigenvalues which appear in the Y shaped branch in Fig. 4.2 correspond to the viscous modes. The discrete modes observed on top of the viscous modes with $c_i \sim 0$ are known as the

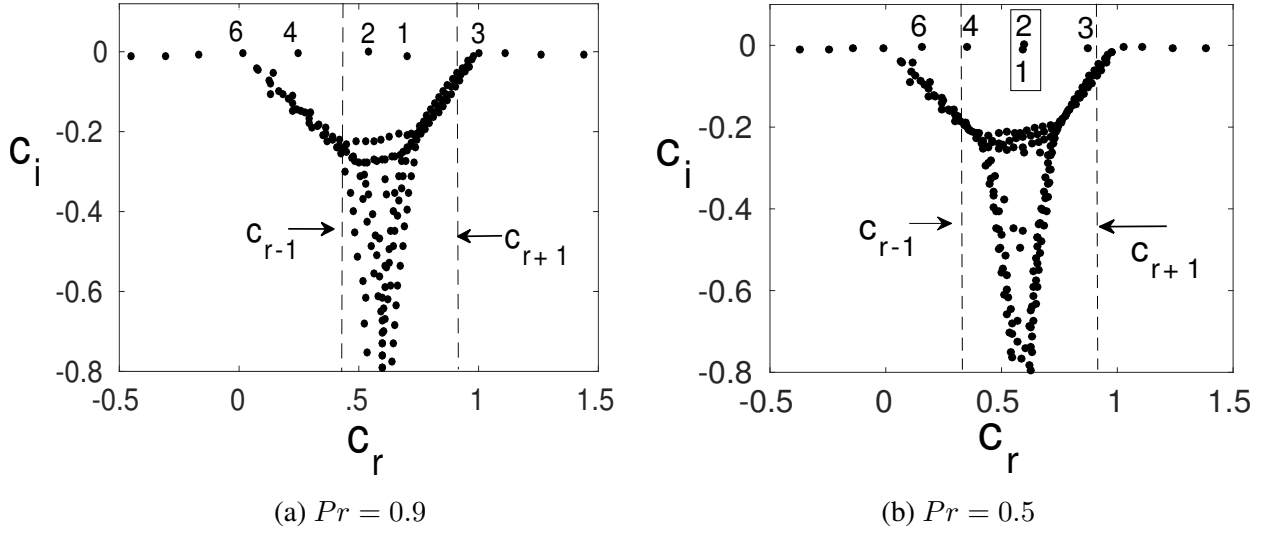


Figure 4.2: As the Prandtl number is reduced from 0.9 to 0.5, eigenvalues of the odd and even acoustic modes approach each other, leading to the synchronization between the modes 1 and 2 (marked by a rectangular box). The parameters considered here are: $M = 12$, $Re = 10^5$, $\beta = 0$ and $\alpha = 5$. The two dashed lines encompass the acoustic modes with two sonic lines.

acoustic modes, and the first few are numbered for reference. As the Reynolds number or the disturbance wave increases, it becomes difficult to capture the Y-spectrum accurately. A minute error in the linear stability operator can affect the eigenvalues that appear in the triple point of the H and Y shaped structure. Although the acoustic modes converge quickly, to resolve the viscous modes we need ample Chebyshev points (~ 500 grid points).

As the Prandtl number is reduced from 0.9 to 0.5, the phase speed of the odd acoustic modes decreases, while it increases significantly for the even modes. A synchronization between the first two acoustic modes is observed when their phase speeds are comparable (identified by the rectangular box in Fig. 4.2b). It is found that such synchronizations can result in high growth rates, which often lead to dominant instabilities in the flow.

Synchronization between the acoustic modes has been presented in [69] for high Mach number flat plate boundary layer flows. They observe two types of discrete modes in the boundary layer – fast and slow modes, and when these modes synchronize, a branching of the discrete spectrum is observed. It is shown that the eigenvalues attain complex conjugate values around the synchronization point, and the eigenmodes exhibit mixed characteristics of both slow and fast modes. We see similar trends in the eigenvalues and eigenvectors in compressible Couette flow, and the branching patterns observed for different values of Prandtl numbers are described below.

4.2.1 Branching pattern at high Prandtl numbers

Fig. 4.3a plots the variation of phase speed and the growth rate with wavenumber for the first four acoustic modes at $M = 12$, $Re = 10^5$ and $Pr = 0.5$. The wavenumber is varied from around 4.5 to 9.6 and the arrows in the figure identify the direction of increasing wavenumber. When the wavenumber is around 4.8, we observe the first synchronization between the first two acoustic modes around a phase speed of 0.6. A magnified view of this synchronization is presented in Fig. 4.3b. When the wavenumber is increased progressively, these two modes cross one another, and they encounter the higher modes at a comparatively larger wavenumber. Synchronizations between the modes 1 and 4, and 2 and 3 are also observed in the range of wavenumbers considered. We notice local peaks and dips in the amplification rate during these synchronizations.

The branching pattern observed during synchronization between the modes 1 and 2 is presented in Figs. 4.3c and 4.3d. The phase speed plotted as a function of the disturbance wavenumber shows a monotonic decrease for mode 1. The phase speed of mode 2 increases continuously, and they attain matching values at a wavenumber of 4.77. Synchronization between the modes makes mode 2 more unstable, and mode 1 gets highly stabilized. We term this branching pattern as “cross-over” branching and is observed in the compressible Couette flow at relatively high Prandtl numbers.

To track the modes accurately when they undergo synchronization, we have to take small steps in disturbance wavenumber as the phase speed of the two modes approaches one another. This is due to the fact that the modes cannot be identified from their eigenfunction since they become identical at the synchronization point.

For the compressible boundary layer flows considered by [69], the branch points are plotted in the complex wavenumber plane (see Fig. 15 in their work). We perform a similar spatio-temporal study for the present flow in the vicinity of mode synchronization. Since all the branch points cannot be located using temporal stability analysis in the real wavenumber plane, we have to use spatio-temporal analysis in the complex wave number plane. Further details can be found in [69]. Though Couette flow considered in this work is x-homogeneous, a disturbance can grow or decay as it is convected downstream. This corresponds to the imaginary part of the wavenumber being negative ($\alpha_i < 0$) or positive ($\alpha_i > 0$), respectively. When the branch points

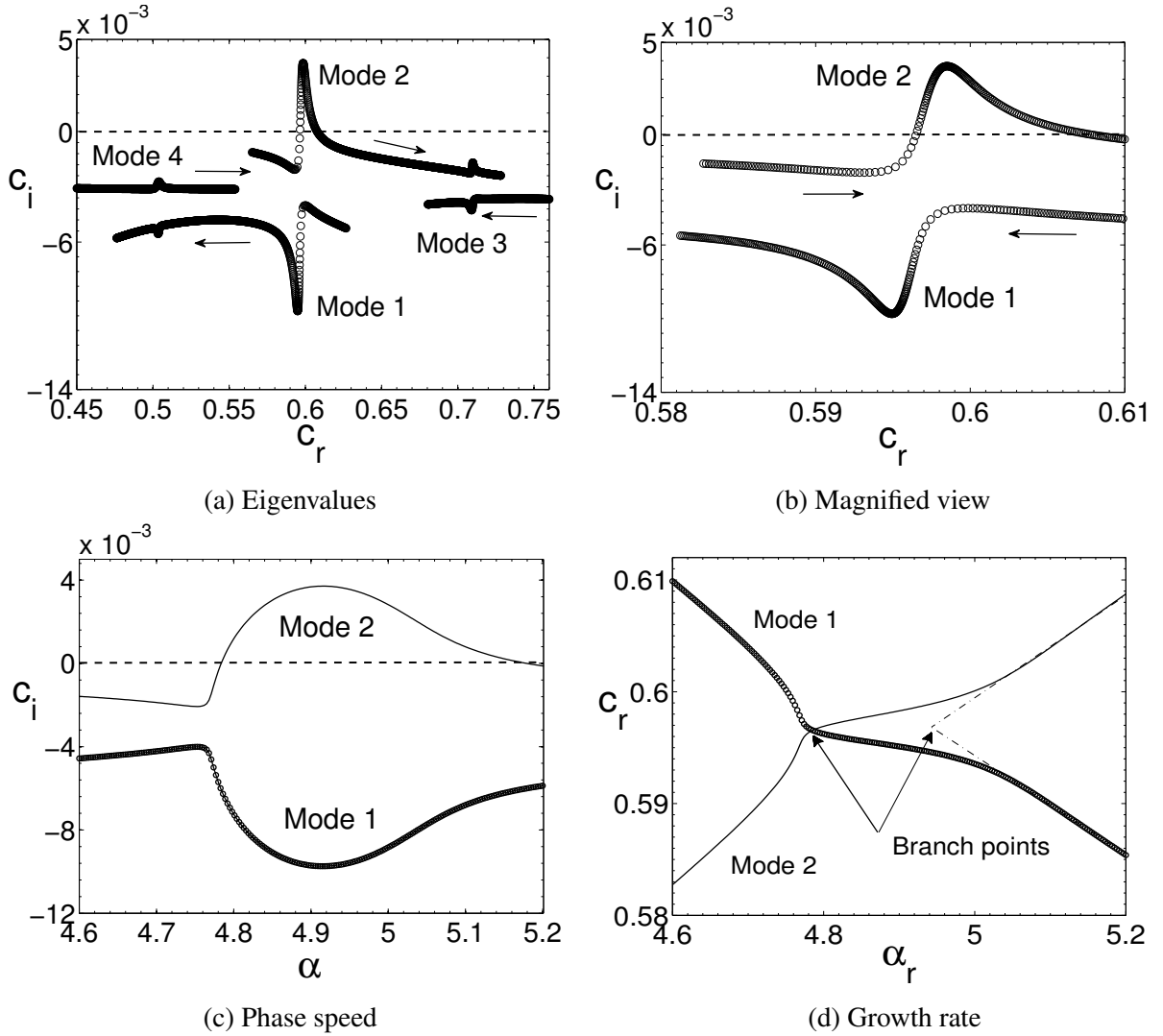
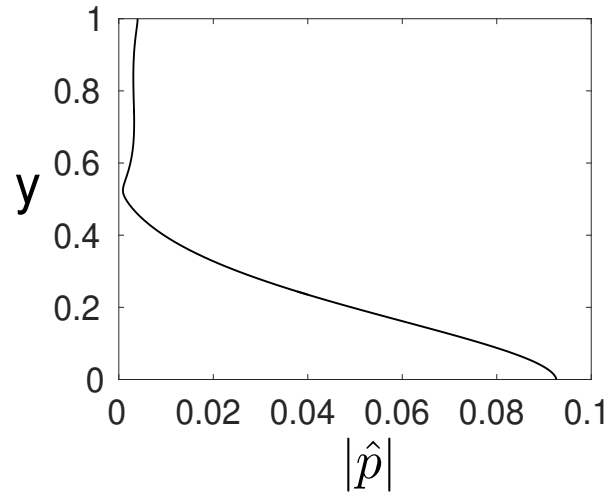


Figure 4.3: Synchronization of the first few acoustic modes for $M = 12$, $Re = 10^5$ and $Pr = 0.5$, characterized by matching phase speeds and local peaks/dips in growth rates. Arrows indicate the direction of increasing wavenumber, and the data in (d) is extrapolated back in α_r to locate the second branch point.

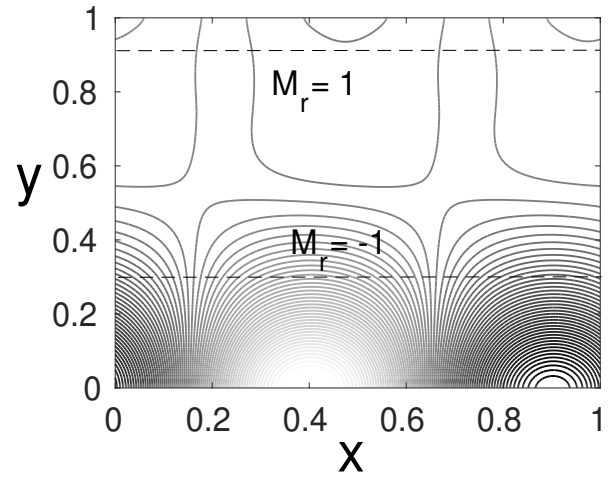
are located in the lower half plane ($\alpha_i < 0$), the solution marching along the real wavenumber axis bypasses the branch points from above. This is identified as branching pattern **B** in [69] and is identical to the cross-over branching discussed above. The two branching points for this case are located at $\alpha_r = 4.77$, $\alpha_i = -0.005$ and $\alpha_r = 4.94$, $\alpha_i = -0.179$ respectively (see Fig. 4.3d). The synchronization pattern between higher modes (1 and 4, and 2 and 3) also exhibit the cross-over branching pattern, similar to the one between the first two acoustic modes.

Eigenfunctions prior to and at synchronization

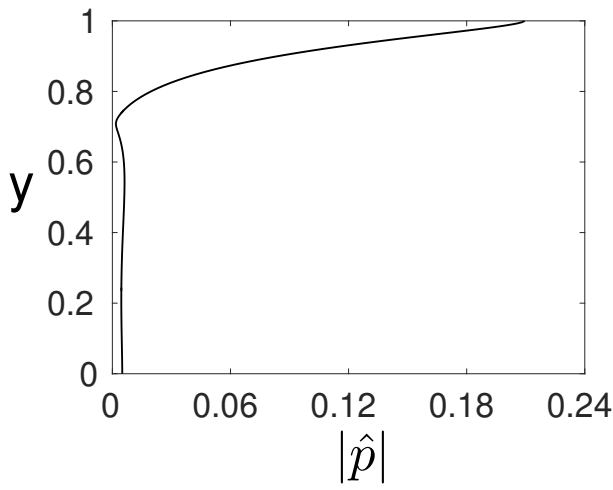
As reported in [20], the peak in the pressure eigenfunction appears close to the top wall for even acoustic modes, and at the bottom wall for the odd modes. It is noted that high-amplitude



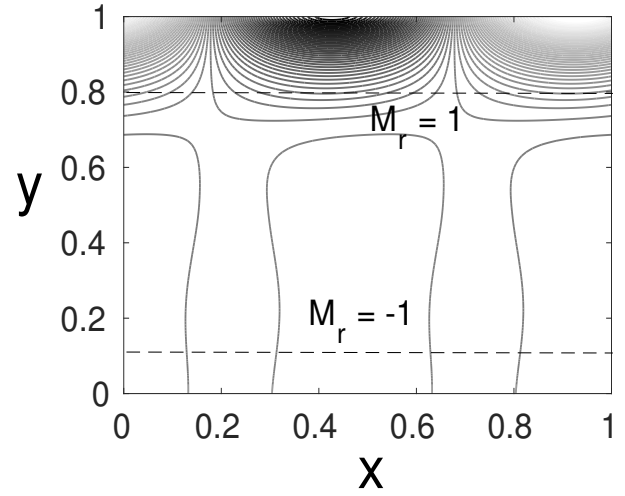
(a) Mode 1



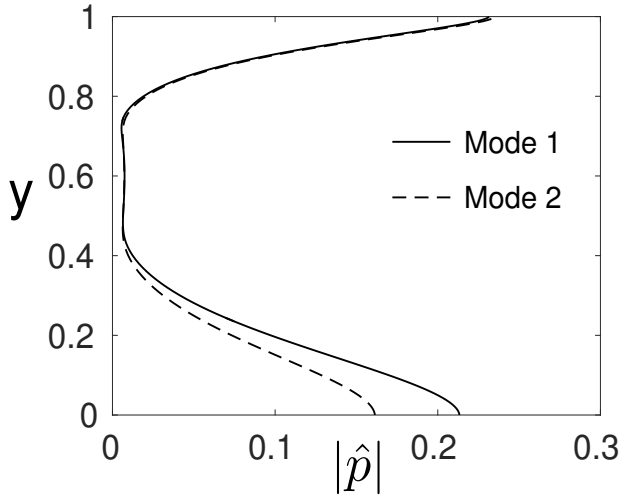
(b) Mode 1



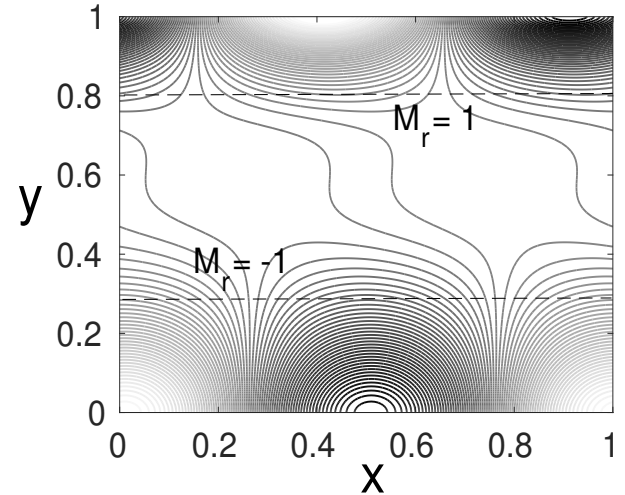
(c) Mode 2



(d) Mode 2



(e) Mode 1 and 2



(f) Mode 1 and 2

Figure 4.4: Pressure eigenfunctions plotted along the shear layer exhibit higher amplitudes in the regions of relative supersonic flow. The parameters used here to plot mode 1 [in (a) and (b)] and 2 [in (c) and (d)] are same as that of Fig. 4.2a. The characteristics representing the synchronization between the modes 1 and 2 shown in (e) and (f) are plotted with the parameters used in Fig. 4.2b.

perturbations appear in an embedded region of local supersonic flow relative to the phase speed of the instability wave. The relative Mach number, defined as $M_r = \frac{\bar{U} - c_r}{\sqrt{\bar{T}}}M$, is used to identify the relative supersonic regions for each mode in a given Couette flow.

In general, acoustic modes have only one relative sonic line (if any), with the odd modes having $M_r = -1$, and $M_r = 1$ for the even modes. But in certain cases, for example, modes 1 and 2 in Fig. 4.2a, and the first four acoustic modes in Fig. 4.2b, have two relative sonic lines. The condition for the odd modes to have the second relative sonic line, i.e., $M_r = 1$ is that the phase speed of the disturbance should satisfy $c_r \leq \bar{U} - \frac{\sqrt{\bar{T}}}{M}$, at the upper wall. Since the values of $\bar{U}$ and $\bar{T}$ are 1 at the upper wall, the phase speed of the limiting disturbance becomes $c_{r+1} = 1 - \frac{1}{M}$. This gives an upper bound on the phase speed of the odd modes to have two relative sonic lines and is marked by a vertical dashed line in Fig. 4.2. Similarly, for the even modes to have the second relative sonic line ($M_r = -1$), they must satisfy $c_r \geq \bar{U} + \frac{\sqrt{\bar{T}}}{M}$, at the lower wall, which yields, $c_{r-1} = \sqrt{\frac{1}{M^2} + \frac{(\gamma - 1)Pr}{2}}$. This expression gives a lower bound on the phase speed of the even modes to exhibit two relative sonic lines, and c_{r-1} is marked by a second vertical dashed line in Fig. 4.2.

The disturbance phase speed c_{r+1} corresponding to $M_r = 1$ for the odd modes is only a function of the reference Mach number. Therefore, when we decrease the Prandtl number from 0.9 to 0.5 (refer Fig. 4.2), c_{r+1} remains constant at 0.9167. However, the phase speed c_{r-1} corresponding to $M_r = -1$ for the even modes is a function of the reference Mach number and the Prandtl number. Its value decreases from 0.4324 to 0.327 as the Prandtl number is reduced. Thus, the range of wave speeds bounded by the two limiting values increase with decreasing Prandtl number and more acoustic modes are observed with two relative sonic lines at lower Prandtl numbers compared to a higher Prandtl number case (compare Figs. 4.2a and 4.2b).

The location of relative sonic lines along with the pressure eigenfunctions for the first two acoustic modes in Fig. 4.2a are shown in Figs. 4.4a and 4.4b and in Figs. 4.4c and 4.4d. We observe that for mode 1, the amplitude of the pressure eigenfunction increases in the relative supersonic region close to the bottom wall, bounded by the sonic line $M_r = -1$, and reaches a maximum value at the bottom wall (see Figs. 4.4a and 4.4b). On the other hand, for mode 2, an increase in the pressure eigenfunction is noticed in the relative supersonic region close to the top wall, bounded by the other sonic line $M_r = 1$, and the maximum amplitude is attained

at the top wall (see Figs. 4.4c and 4.4d). The pressure eigenfunctions corresponding to the synchronization between the first two acoustic modes are shown in Figs. 4.4e and 4.4f. In this case, high amplitude pressure eigenfunctions are observed near both the walls, in the regions bounded by the respective sonic lines. The variation of the eigenfunctions across the shear layer is almost identical for both the modes when their phase speeds are close. Similar patterns for the eigenfunctions during the synchronization between the fast and slow modes in a high-speed flat plate boundary layer flow are reported in [69] (see Figs. 23 and 24 in their paper).

4.2.2 Branching pattern at low Prandtl numbers

Fig. 4.5a presents the synchronization pattern obtained at $M = 12$, $Re = 10^5$ and a Prandtl number of 0.4. We plot the phase speed and growth rates of the first four acoustic modes and once again, track their variation with increasing wavenumber. High growth rates are observed at the synchronization points similar to that shown in Fig. 4.3a. However, we notice a prominent difference in the branching pattern corresponding to the synchronization between the acoustic modes 1 and 2 as compared to the higher Prandtl number ($= 0.5$) case. A magnified view is presented in Fig. 4.5b as these two modes approach each other when the wavenumber is increased. The closest phase speed between the modes is attained at $\alpha_r = 4.19$, beyond which they revert back and move away. Modes 1 and 2 do not cross over as seen in Fig. 4.3b.

We refer to the branching pattern at low Prandtl numbers as the “revert-back” branching, where the phase speeds of the two modes show non-monotonic behavior with increasing wave number. Mode 1, as expected, has a higher phase speed than mode 2, initially. After synchronization, the phase speed of mode 1 increases again with an increase in the wave number to attain values higher than mode 2 (see Fig. 4.5c). This is opposite to the trend seen in the “cross-over” branching at higher Prandtl number (Fig. 4.3c). Also, the growth rate of mode 1 shows a sharp increase during synchronization and there is a corresponding decrease in the growth rate of mode 2 (see Fig. 4.5d). On the contrary, mode 2 is destabilized due to synchronization with mode 1 (see Fig. 4.3d) at higher Prandtl numbers.

The variation of phase speed plotted as a function of wavenumber (Fig. 4.5c) shows that the phase speed of the first two acoustic modes does not match for any real wavenumber value. However, a branch point is observed in the complex wavenumber plane correspond-

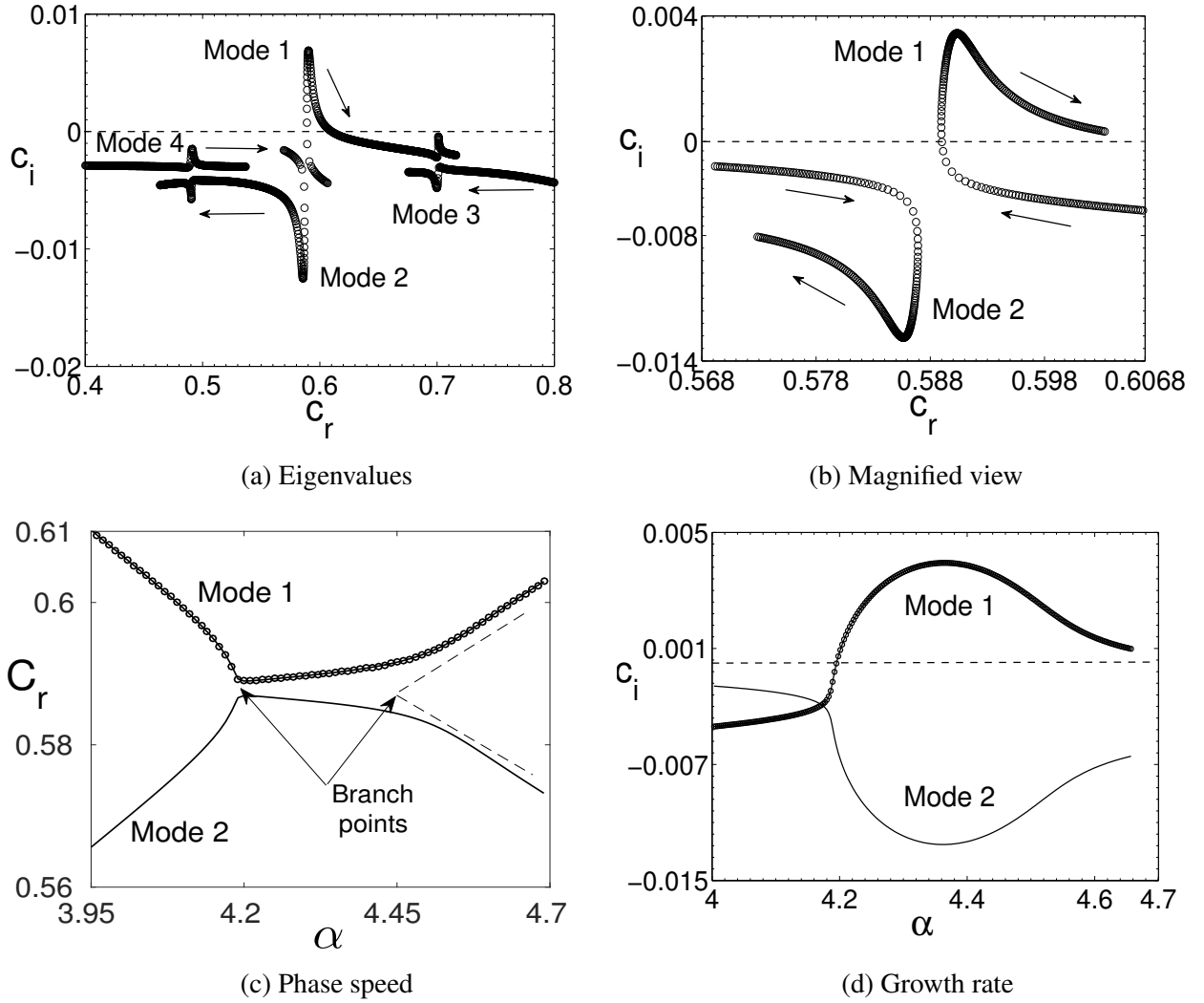


Figure 4.5: Variation of growth rate and phase speed with wavenumber for the first few acoustic modes at $M = 12$, $Re = 10^5$ and $Pr = 0.4$. Modes 1 and 2 do not attain a common phase speed, but the higher modes 3 and 4 synchronize with modes 1 and 2 respectively. Increase in wavenumber is shown by the direction of arrows.

ing to $\alpha_r = 4.19$ and $\alpha_i \simeq 0.0044$. A second synchronization point is obtained at $\alpha_r = 4.45$ and $\alpha_i \simeq -0.115$ by extrapolating back in α_r , as shown in Fig. 4.5c. As we march along the real wavenumber axis, the first synchronization point ($\alpha_i > 0$) has to be bypassed from below, and the second synchronization point ($\alpha_i < 0$) is bypassed from above. This is identified as the branching pattern C in [69].

When the modes revert back as shown in Fig. 4.5b, they exchange their characteristics. We notice that peaks in the eigenfunctions appear near the top and the bottom wall for the modes 1 and 2 respectively, which is opposite to that observed originally in Figs. 4.4a and 4.4b before synchronization. This is in contrast to the high Prandtl number case ($= 0.5$), where modes 1 and 2 retain their characteristics past the synchronization point in the cross-over branching.

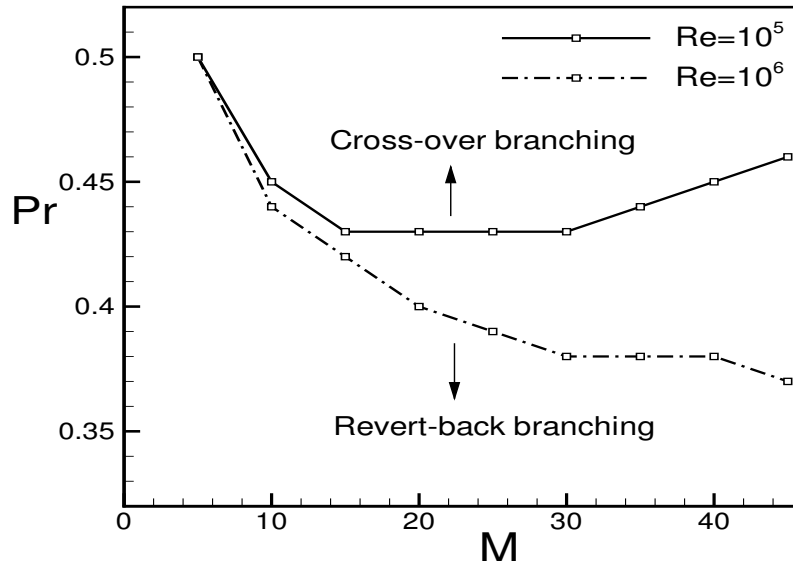


Figure 4.6: The threshold Prandtl number is shown as a function of Mach number at two different Reynolds numbers $Re = 10^5$ and $Re = 10^6$. It separates the two branching patterns, i.e., “cross-over” (at high Pr) and “revert-back” (at low Pr). Data here corresponds to the synchronization between the acoustic modes 1 and 2.

4.2.3 Effect of Prandtl number on the branching pattern

For a fixed Mach number and Reynolds number, there exists a distinct Prandtl number that distinguishes the region where modes 1 and 2 cross one another or revert back with increasing wavenumber. This is illustrated in Fig. 4.6, which plots the threshold Prandtl number as a function of Mach number, at Reynolds numbers of 10^5 and 10^6 . When the Prandtl number is higher than the threshold, “cross-over” branching pattern is observed corresponding to the synchronization between the first two acoustic modes. The “revert back” branching pattern is noticed for Pr values lower than the threshold. The threshold Prandtl number is found to decrease for higher Reynolds number at high Mach numbers.

As noted earlier, “cross-over” branching between the first two acoustic modes is associated with unstable mode 2 disturbances. On the other hand, mode 1 is destabilized when the acoustic modes undergo a “revert-back” branching. Thus, the threshold Prandtl number for a given Mach and Reynolds number also distinguishes the parameter values where mode 1 or mode 2 will be the dominant instability. However, determining the Prandtl number which separates one branching pattern from another, when the dispersion curve branches out, for a range of Mach number and Reynolds number can be a manually intensive process. Identification of modes causing instability is discussed in detail in the next section.

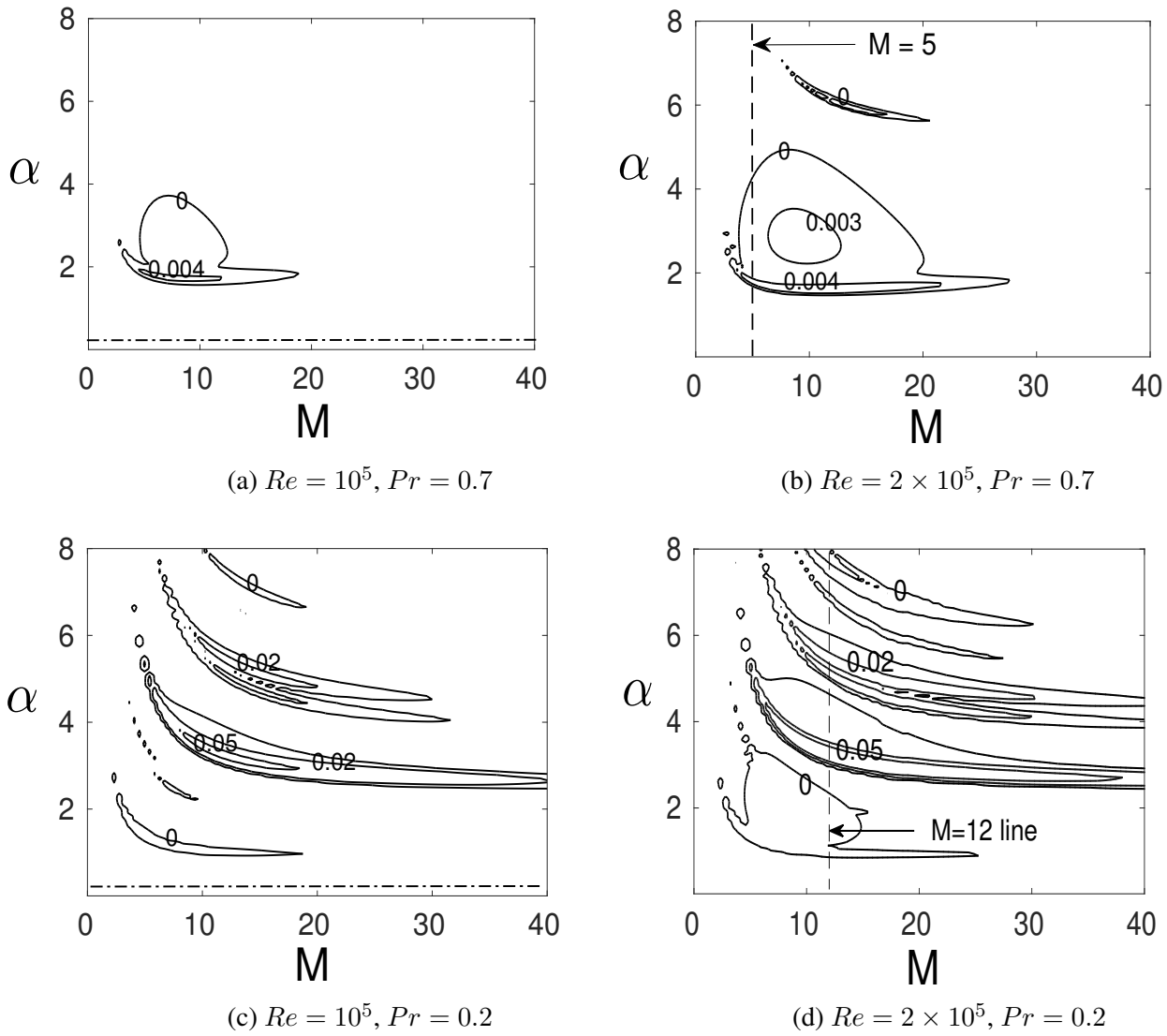


Figure 4.7: Stability diagrams in the $M - \alpha$ plane show that as the Prandtl number is reduced, growth rates, the range of instability in Mach number as well as the number of instability loops increase for all the Reynolds number cases considered.

4.2.4 Stability diagrams in the $M - \alpha$ plane

Fig. 4.7 presents the stability diagram in the Mach number versus wavenumber plane for two-dimensional disturbances at various Reynolds numbers and Prandtl numbers. The growth rate corresponding to the most unstable mode is plotted and determining the amplification rates for a range of Mach number and wavenumber can be a computationally intensive process. The regions within the loops in Fig. 4.7 are unstable with positive growth rates and the region outside is stable. For the two Reynolds numbers considered, it can be observed that as the Prandtl number decreases for a given Re , the region of instability, as well as the magnitude of the growth rates increase. For instance, at $Re = 10^5$, as the Prandtl number reduces from 0.7 (Fig. 4.7a) to 0.2 (Fig. 4.7c) there are additional loops of instability with higher growth rates. The maximum

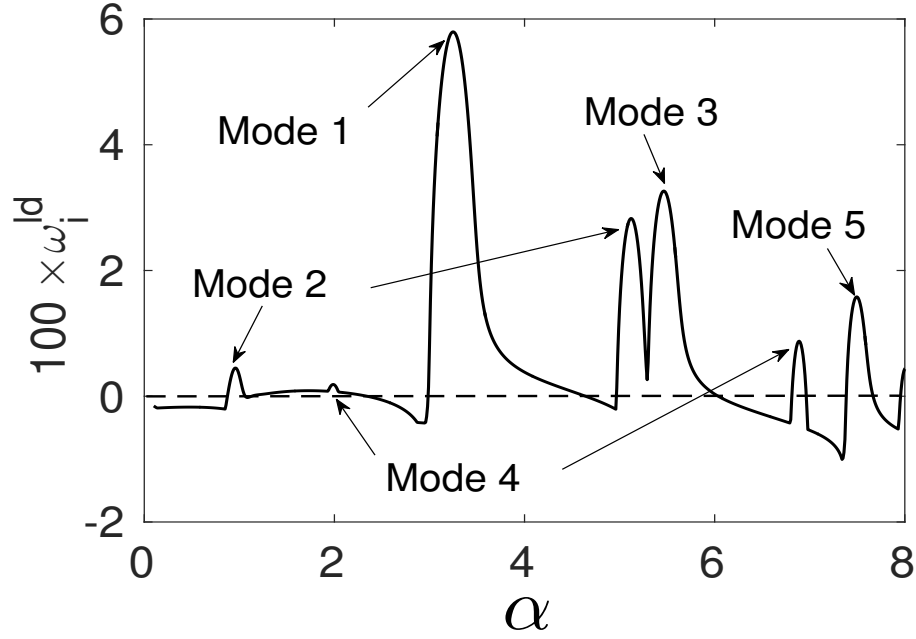


Figure 4.8: The dominant unstable acoustic modes at various wave numbers for a Couette flow at $M = 12$, $Pr = 0.2$, and $Re = 2 \times 10^5$ are identified. The instability peaks correspond to synchronizations between acoustic modes.

growth rate increases from 0.004 to 0.05 with a decrease in the Prandtl number value. Hence, for a fixed Reynolds number, the effect of reducing the Prandtl number is destabilizing.

For determining the effect of Reynolds number at a fixed Prandtl number, Figs. 4.7a and 4.7b are compared. When the Reynolds number increases from 10^5 to 2×10^5 at $Pr = 0.7$, it can be noticed that the size of the instability loops increases, and additional loops of instability are seen at this comparatively high value of Prandtl number. Similar effects are also observed at 0.2 (compare Figs. 4.7c and 4.7d). Ref. [22] presents comparable results for $Pr = 0.72$, at the same Reynolds numbers (10^5 and 2×10^5).

Identification of the modes causing instability

A high Reynolds number and low Prandtl number lead to large regions of instability in the $M - \alpha$ plane. In order to identify the acoustic modes associated with the various instability loops, we study the case of $Re = 2 \times 10^5$ and $Pr = 0.2$ (Fig. 4.7d). Several instability loops are observed in the Mach number range of 5 – 40. For Mach numbers beyond 40, the number of instability loops decreases. The only exception is the second loop with a maximum growth rate around 0.02 well into the high Mach number range. This is of particular interest because the low value of Prandtl number considered here is expected to occur at low pressure and high-

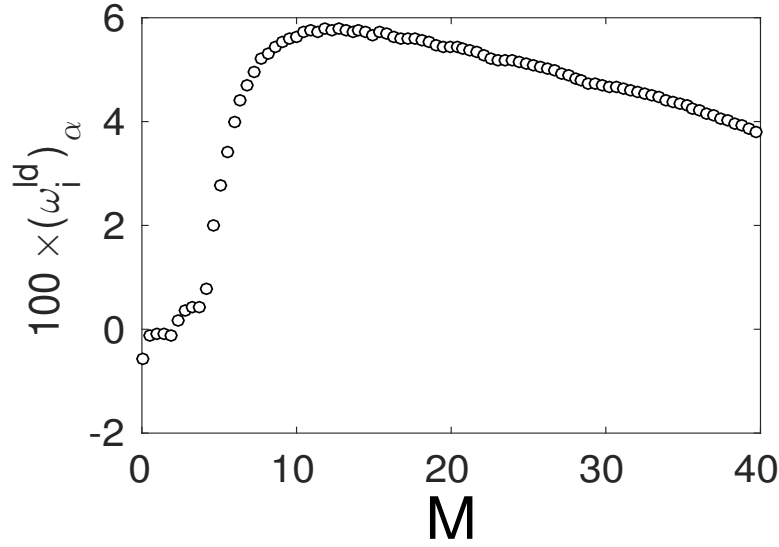


Figure 4.9: Maximum growth rate over a range of wavenumbers (0 – 8) is plotted as a function of M at $Re = 2 \times 10^5$ and $Pr = 0.2$. Mode 1 is the most dominant unstable mode for Mach numbers exceeding 5.

temperature conditions, which are encountered during re-entry flight at high Mach number and high altitude.

The variation of the growth rate of the least stable mode (ω_i^{ld}) with wave number for a fixed Mach number of 12 is plotted in Fig. 4.8. The values of Reynolds number and Prandtl number match those in Fig. 4.7d, and the data is extracted along the dashed line shown in the figure. The highest growth rate is found at a wavenumber of 3.25, corresponding to the second loop in Fig. 4.7d. This is caused by acoustic mode 1, with a growth rate of around 0.058. The instability peaks due to mode 2 appear at two wavenumbers, namely, 0.96 (lowest loop in Fig. 4.7d) and 5 (third instability loop in Fig. 4.7d). Acoustic modes 3, 4 and 5 also appear as dominant instabilities as discrete peaks around $\alpha = 5.47$, $\alpha = 1.99$ and 6.89 , and $\alpha = 7.52$ respectively.

We note that the instability peaks listed above correspond to mode-synchronizations between various acoustic modes. For example, the high growth rate at $\alpha = 3.25$ is due to the synchronization of modes 1 and 2, which is akin to the case presented in Fig. 4.5. Similarly, the growth rate peak observed around $\alpha = 5$ is due to the synchronization between the second and the fourth acoustic modes.

Fig. 4.9 presents the variation of maximum growth rate over a range (0–8) of wave numbers for a $Re = 2 \times 10^5$, $\beta = 0$ and $Pr = 0.2$ Couette flow. Mode 1 is the most dominant instability for most of the range of Mach numbers (≥ 5). This is due to the revert back branching on synchronization of modes 1 and 2 (refer section 4.2.2). Mode 2 is the most unstable mode in

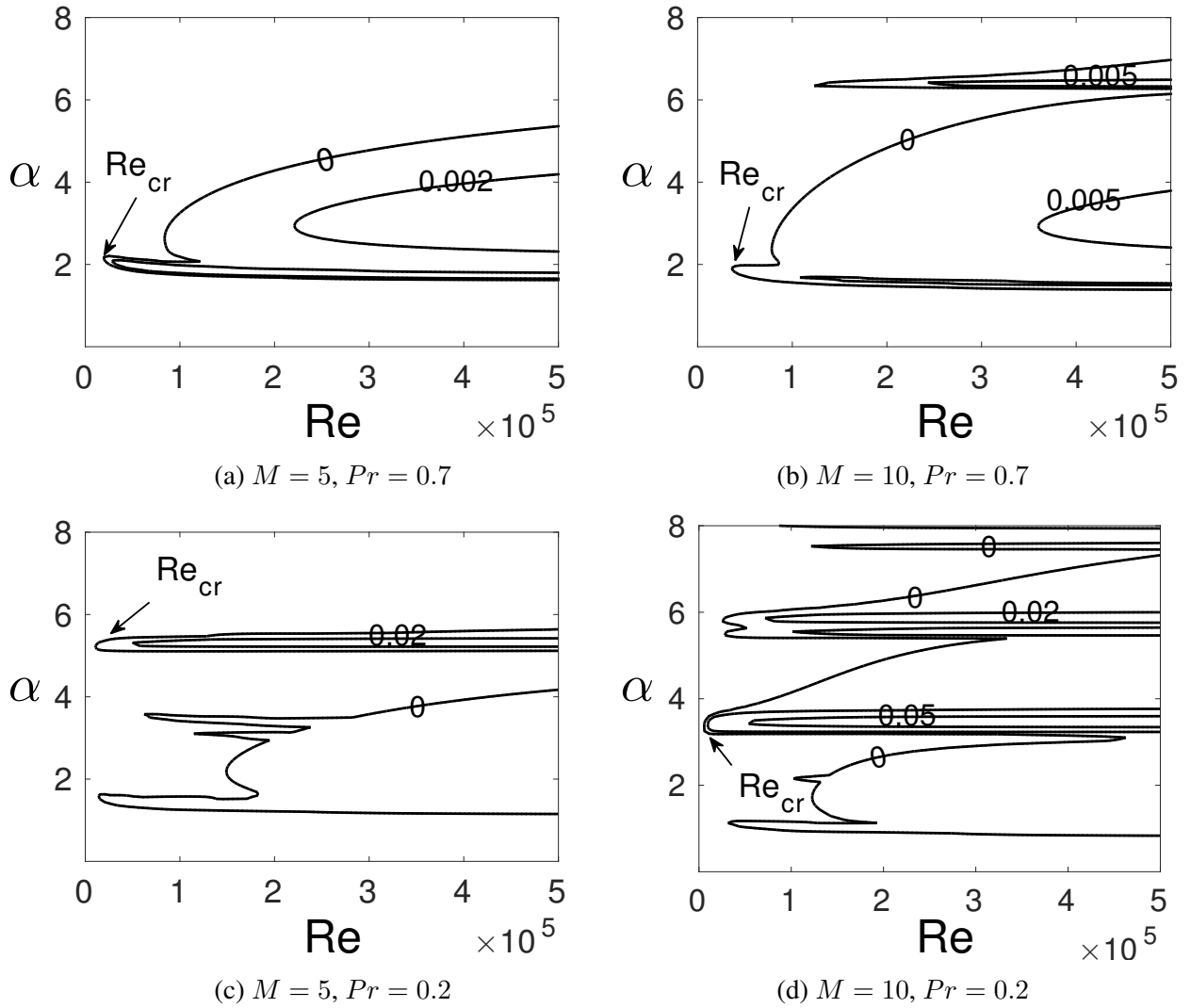


Figure 4.10: Stability diagrams in the $Re - \alpha$ plane plotted for two different Mach numbers show that, as Pr is reduced critical Reynolds number also decreases.

the low Mach number range ($M < 5$) and comes from the bottom instability loop of Fig. 4.7d. On the other hand, at high Prandtl numbers, for example, at $Pr = 0.72$, mode 2 dominates over the entire range of Mach number ($0.1 - 40$). In this case, the synchronization between the first two acoustic modes makes mode 2 more unstable, due to the cross-over branching explained in Section 4.2.1. Ref. [22] reports similar findings for a uniform viscosity and conductivity Couette flow at $Pr = 0.72$.

4.2.5 Stability diagrams in the $Re - \alpha$ plane

Fig. 4.10 presents the stability diagram in the Reynolds number versus wavenumber plane for two Mach numbers 5 and 10, with different values of Prandtl numbers. For a given Mach number, as Prandtl number is lowered, more instability loops appear in addition to the bottom loop. Also, the peak growth rate increases with the reduction in the Prandtl number value. This

is more prominent when we compare different Pr cases at $M = 10$. For example, as the Prandtl number is decreased from 0.7 to 0.2, the peak growth rate increases from 0.005 to 0.05. When the two Mach number (5 and 10) cases are compared for a fixed Prandtl number, it can be seen that the peak growth rate increases with increase in Mach number.

The critical Reynolds numbers (Re_{cr}) along with the critical wave numbers (α_{cr}) are listed in Table 4.1. It can be noted that, for both $M = 5$ and $M = 10$, critical Reynolds number decreases monotonically as the Prandtl number is reduced. However, the variation of critical Reynolds number is not monotonic with Mach number. This is also observed in [22] for a Couette flow with both uniform as well as non-uniform shear. For Prandtl numbers of 0.7 and 0.5, the critical Reynolds number increases as the Mach number is increased. However, when Pr reaches a value of 0.2, the critical Reynolds number attains a lower value at $M = 10$ compared to $M = 5$. Therefore Mach number has a dual role of stabilizing and destabilizing the flow, depending on the Prandtl number.

4.2.6 Effect of heating/cooling the lower wall

The destabilizing effect of wall cooling has been reported earlier; see the works of [62], [63] and [70]. In this section, we examine the effect of variation in Prandtl number at different wall cooling/heating conditions. For this, we choose a parameter r , which is the ratio of bottom wall temperature (T_w) to the adiabatic wall temperature (T_{ad}). The adiabatic wall temperature is a function of the Mach number and the Prandtl number and is defined as, $T_{ad} = 1 + \frac{(\gamma - 1)}{2} Pr M^2$. The lower wall is defined as cold or hot if the temperature is less than or greater than T_{ad} respectively. $r = 1$ represents the case of an adiabatic bottom wall.

Fig. 4.11 shows the variation of the maximum growth rate in the $r - \alpha$ plane for $M = 5$ and $Re = 2 \times 10^5$ case, at different Prandtl numbers. We notice that lowering the bottom

Table 4.1: Critical Reynolds number and corresponding wave number for a uniform viscosity and conductivity plane Couette flow at varying Prandtl numbers and for two Mach numbers.

Pr	M=5		M=10	
	Re_{cr}	α_{cr}	Re_{cr}	α_{cr}
0.7	19036	2.16	35680	1.90
0.5	15136	1.98	30042	1.66
0.2	10442	5.22	5232	3.34

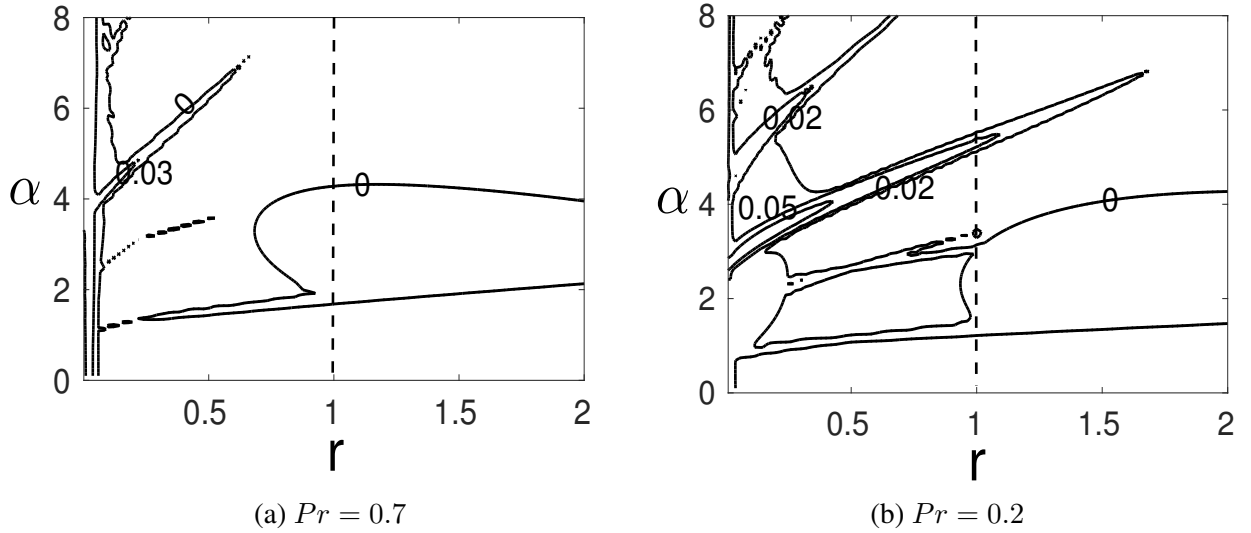


Figure 4.11: Growth rate contours plotted as a function of wall cooling parameter ($= r$) and disturbance wavenumber at $M = 5$, $\beta = 0$ and $Re = 2 \times 10^5$. Data shows that wall cooling increases the regions of instability with higher growth rates, and reducing the Prandtl number increases the maximum growth rate in each loop.

wall temperature leads to higher growth rates for both the Prandtl numbers, while heating has a mild stabilizing effect. It can be observed that when cooling increases (r decreases), additional instability loops appear, and the maximum growth rate also increases. Also, the location of the peaks shifts to a lower wave number, as cooling is increased.

Loops and spikes in Fig. 4.11 conform to different instability loops in the corresponding $M - \alpha$ diagram. For instance, the growth rates observed along $r = 1$ line in Figs. 4.11a and 4.11b correspond to the growth rates along a line at $M = 5$ in Figs. 4.7b and 4.7d respectively. The values of wavenumber ranging from 1 – 4 at $r = 1$ in Fig. 4.11b corresponds to the lower instability loop in Fig. 4.7d. Also, the spike that appears around $\alpha = 5$ and $r = 1$ in Fig. 4.11b corresponds to the second instability loop of Fig. 4.7d with a growth rate around 0.02.

4.2.7 Effect of three-dimensional perturbation

Fig. 4.12 compares the amplification rate of the most unstable mode for three-dimensional disturbances ($\beta \neq 0$) with a two-dimensional perturbation ($\beta = 0$) at $Re = 2 \times 10^5$ and $Pr = 0.5$. Figs. 4.12a and 4.12b correspond to $M = 5$ and 10 respectively. For both the Mach number cases, it is observed that the growth rates of the three-dimensional perturbations are lower than their two-dimensional counterpart for most of the range of wave number. Also, the peak growth rates shift slightly to higher wave numbers as β is increased. Fig. 4.12a shows that at some values of wave number, such as, α from 5.98 – 7.19, three-dimensional disturbance ($\beta = 0.8$

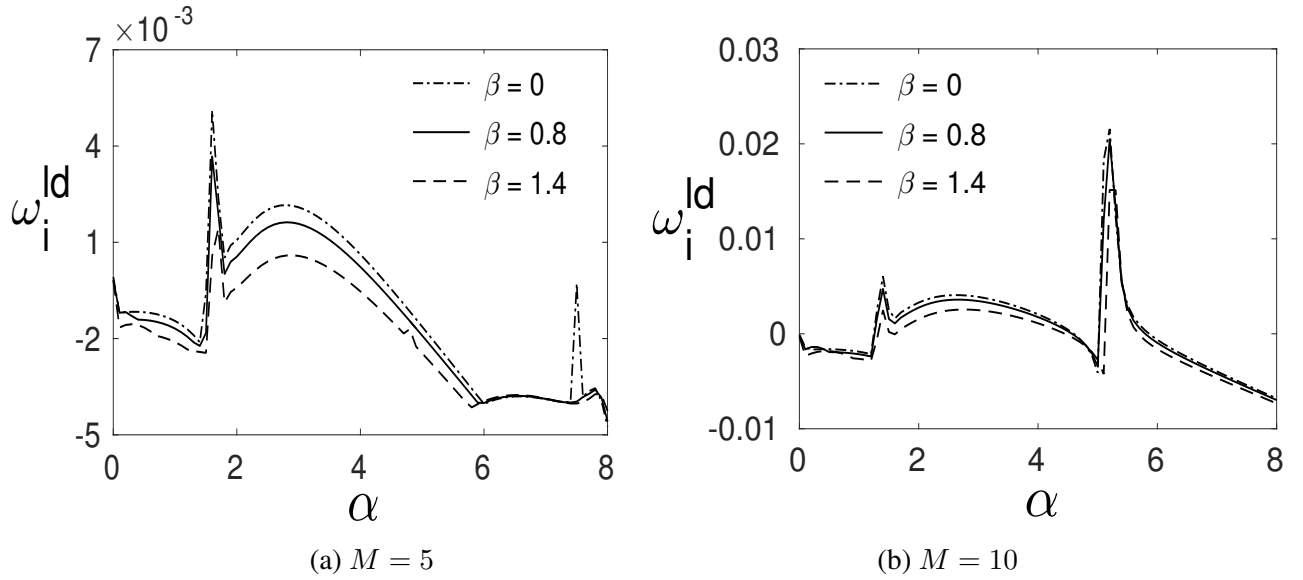


Figure 4.12: Comparison of the growth rate of a two-dimensional perturbation with three-dimensional perturbations as a function of wavenumber at $Re = 2 \times 10^5$ and $Pr = 0.5$. The growth rate of the two-dimensional disturbance is higher than the three-dimensional perturbation for most of the range of wavenumbers considered.

and 1.4) have higher growth rates compared to two-dimensional perturbation. Similar trends are observed at a higher Mach number of 10 (Fig. 4.12b). Ref. [22] also investigated the effect of three-dimensional perturbations on the least stable growth rate for different spanwise wave numbers, with parameters $M = 15$ and $Re = 10^5$. They found that for a certain range of wave number, three-dimensional disturbances can be more unstable than their two-dimensional counterpart for a uniform shear flow.

4.2.8 Transient growth analysis

Fig. 4.13 presents the maximum temporal energy amplification (G_{max}) as a function of the streamwise and spanwise wavenumber at $M = 10$. The range of streamwise wavenumbers are selected such that there does not exist any unstable mode in this region and it is shown by a dashed line in Figs. 4.7a and 4.7c. When the waves are three-dimensional, their growth rate decrease significantly (see Fig. 4.12). Therefore the flow is modally stable in the span of wavenumbers considered in Fig. 4.13.

To investigate the effect of a reduction in the Prandtl number on the short-term energy growth, we compare $Pr = 0.2$ flow in Fig. 4.13c to flows with $Pr = 0.4$ (Fig. 4.13b) and $Pr = 0.7$ (Fig. 4.13a) respectively. For the three cases considered here, the maximum amplification of the transient energy occurs for a streamwise independent disturbance ($\alpha = 0$). As the Prandtl

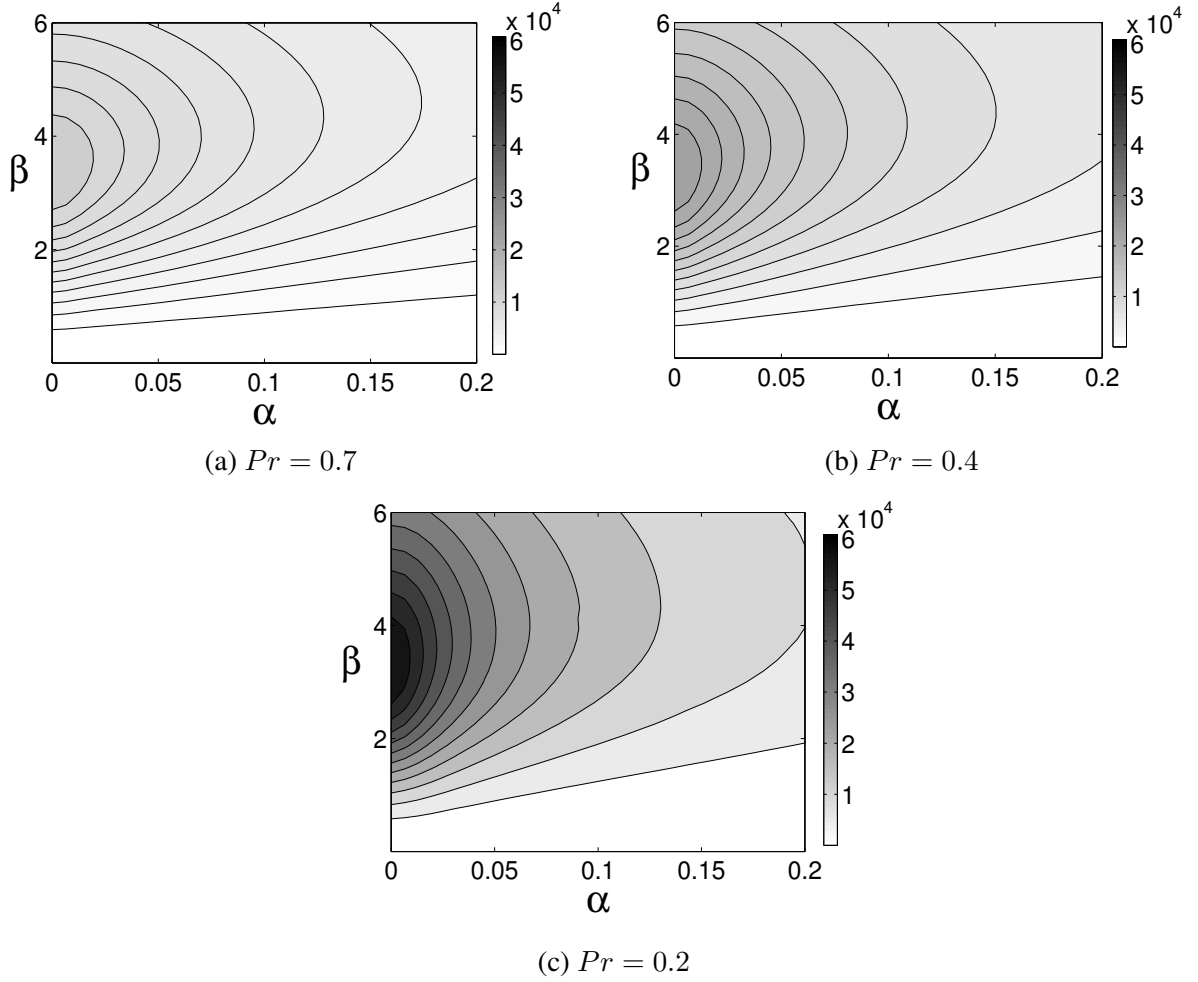


Figure 4.13: Contours of maximum temporal energy amplification G_{max} in the wavenumber plane, showing the effect of a decrease in the Prandtl number value at a Mach number of 10.

number decreases from 0.7 to 0.4, we notice an increase in the energy amplification. A further reduction in the Pr value to 0.2 leads to a significant increase in the disturbance energy at a finite time. Therefore transient growth analysis follows the same trend as that predicted by the linear stability theory, i.e., the strong destabilizing role of decreasing the Prandtl number value.

4.3 Summary

In this chapter, we investigated the effect of Prandtl number on the stability of a uniform viscosity and conductivity Couette flow at high Mach numbers. Prandtl number values are systematically varied from 0.2 – 0.9 by changing the relative magnitudes of viscosity and conductivity, and a temporal linear stability analysis is performed. We find that decreasing the Prandtl number has a destabilizing effect on the flow at all Mach and Reynolds numbers. The critical Reynolds number is significantly reduced, by up to an order of magnitude, especially at high

Mach numbers. The flow is more unstable at lower Prandtl numbers irrespective of the wall boundary conditions (cold/hot/adiabatic). High growth rates with additional instability loops are observed in the stability diagrams as the Prandtl number is decreased.

It is found that the local peaks in the growth rate are due to the synchronization between the acoustic modes. Two types of branching patterns are observed at the synchronization point, namely, cross-over and revert-back. Cross-over branching between the first two acoustic modes occurs at relatively high Prandtl numbers. It is found to destabilize mode 2 acoustic instabilities, and mode shapes are retained through the synchronization process. By comparison, revert-back synchronization is usually noticed at relatively low Prandtl numbers. It results in high mode 1 growth rates, and the even and odd acoustic mode shapes are exchanged past the synchronization point. We compute the threshold Prandtl number that distinguishes the two branching patterns and thus determines the dominant instability mode in a compressible Couette flow.

A non-modal energy analysis is also carried out to investigate the effect of a change in the Prandtl number on the short-term growth of the disturbance. We observe that an increment in the Prandtl number value can significantly reduce the transient energy amplification. The prediction of transient growth analysis is found to be similar to that of the asymptotic results.

Chapter 5

Conductivity stratified flow

In this chapter, we investigate the stability characteristics of a conductivity stratified flow using modal as well as non-modal analysis. Governing equations for the base flow and the perturbations are derived. An energy analysis is also conducted to study the effect of variation in thermal conductivity on different constituents of total perturbation energy.

5.1 Base flow

In a κ -stratified flow, since the viscosity is maintained at its reference value ($=1$), velocity varies linearly as the wall-normal distance ($\bar{U} = y$). The thermal conductivity is varied using Sutherland's formula (Eq. 5.1), where T_{ref} denotes the dimensional reference temperature.

$$\bar{\kappa} = \bar{T}^{3/2} \frac{(1 + B)}{(\bar{T} + B)}, \text{ where } B = \frac{194 \text{ K}}{T_{ref}}. \quad (5.1)$$

Since the lower wall is adiabatic, using Eq. 5.1, the energy equation (Eq. 3.2) can be reduced to

$$\frac{d\bar{T}}{dy} = -(\gamma - 1) M^2 Pr \frac{(\bar{T} + B)}{\bar{T}^{3/2} (1 + B)} y. \quad (5.2)$$

With an initial guess for the temperature at the lower wall, the above equation is solved numerically using a 4th order Runge Kutta method, till the top wall temperature reaches its reference value ($= 1$).

The base flow thus obtained is presented in Fig. 5.1, for a Prandtl number of 0.72 and a reference temperature of 220.8 K. Fig. 5.1a compares the variation in temperature along the wall-normal direction at two Mach numbers 2 and 5. Starting from the same reference quantities

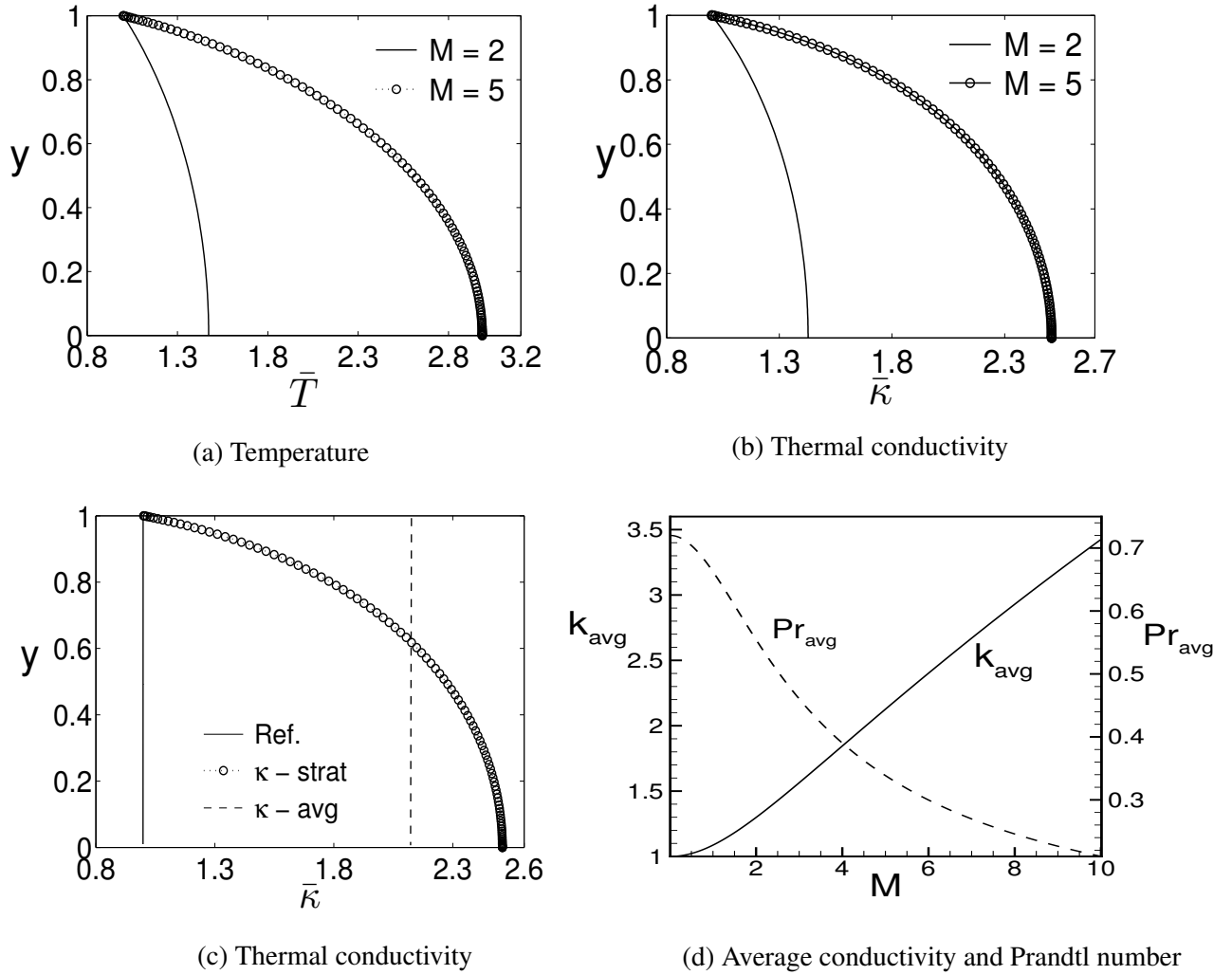


Figure 5.1: The variation of (a) mean temperature and (b) mean conductivity along the wall-normal direction for a κ -stratified flow at two Mach numbers; (c) comparison of conductivity variation in the three model base flows at $M = 5$; (d) average thermal conductivity and Prandtl number for a range of Mach numbers. The reference Prandtl number considered is 0.72.

at the top wall, temperature increases and attains a maximum value at the bottom adiabatic wall. We notice an increment in the mean value of temperature as well as its gradient with an increase in the flow Mach number. As expected, the thermal conductivity exhibits similar variation with wall-normal distance and Mach number (see Fig. 5.1b). The mean conductivity $\bar{\kappa}$ is higher at every point in the κ -stratified flow as compared to the reference flow with constant values of viscosity and thermal conductivity. Therefore, the κ -stratified flow can be viewed as a superposition of an increase in the average value of thermal conductivity, and an imposed conductivity variation across the shear layer. The average value of conductivity is defined as,

$$\bar{\kappa}_{avg} = \int_0^1 \bar{\kappa} dy. \quad (5.3)$$

We construct another model problem with this average value of conductivity and the reference

value of viscosity, both constant across the shear layer, and refer to it as the κ -averaged flow. Fig. 5.1c compares the κ -stratified flow at Mach 5 to the corresponding reference and the κ -averaged flows.

The effect of the gradient or the stratification in conductivity on flow stability can be obtained by comparing the κ -averaged flow with the κ -stratified case. On the other hand, a comparison of the reference flow with the κ -averaged case will bring out the effect of enhanced conductivity. This can be considered as comparing two uniform viscosity, but different Prandtl number flows, resulting from two different values of conductivities. The Prandtl number in the case of the κ -averaged flow is much lower than that of the reference flow and is defined as $Pr_{avg} = Pr/\bar{\kappa}_{avg}$. It can be observed that with an increase in the flow Mach number, κ_{avg} increases sharply, leading to a decrease in the Pr_{avg} value (see Fig. 5.1d).

5.2 Stability results

5.2.1 Linear stability results

Stability plots in Mach number-wavenumber plane

Fig. 5.2 plots the growth rate contours for the least stable mode for varying Mach number and disturbance wave number. A large range of Mach number up to 20 is considered, and the results are representative of high Mach number perfect gas flow. The Reynolds number and reference Prandtl number considered are 10^5 and 0.72 respectively. The regions within the loops are unstable, bounded by the neutral growth rate contours. The stability characteristics of the reference flow with $\bar{\mu} = 1$ and $\bar{\kappa} = 1$ is shown in Fig. 5.2a. The corresponding conductivity-stratified flow is presented in Fig. 5.2d, where viscosity is maintained at its reference value, such that Reynolds number is 10^5 for the entire range of Mach numbers. Fig. 5.2c shows the results for the κ -averaged flows, with uniform conductivity maintained at the $\bar{\kappa}_{avg}$ value, computed for each Mach number. Comparison of Fig. 5.2c and 5.2d thus brings out the isolated effect of conductivity stratification at each Mach number. This is in contrast to earlier works (for example, Ref. [22]), which report the combined effect of the variation in the magnitude and gradient of transport properties on Couette flow stability.

Noting that the value of $\bar{\kappa}_{avg}$ is a strong function of Mach number (see Fig. 5.1d), we present

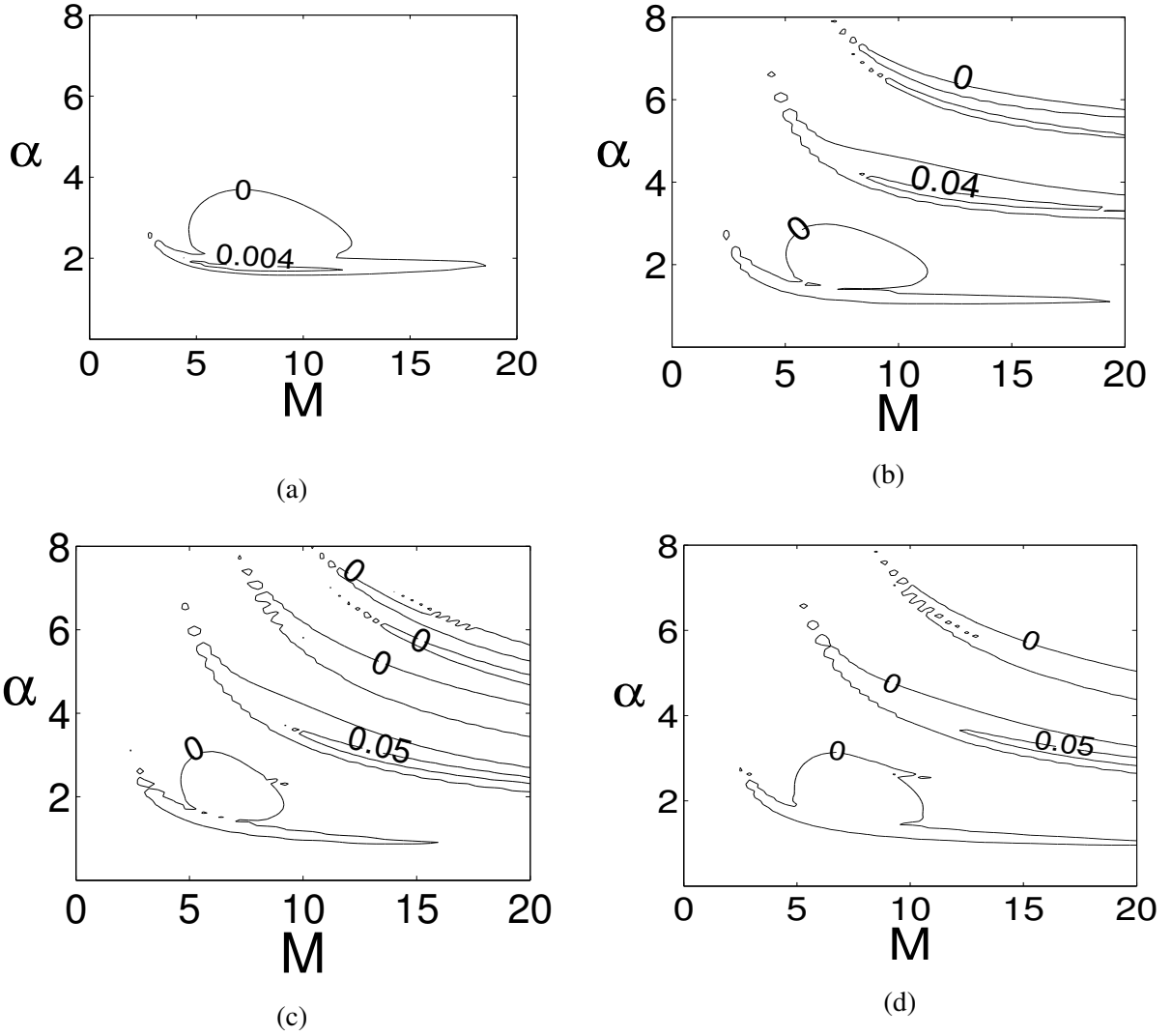


Figure 5.2: Comparison of stability diagrams in the Mach-streamwise wavenumber plane for (a) the reference flow, (b) κ -averaged flow, with $\bar{\kappa}_{avg} = 3.33$; (c) κ -averaged flow with $\bar{\kappa}_{avg}$ varying with Mach number; and (d) κ -stratified flow. Here, reference Prandtl number and Reynolds number are 0.72 and 10^5 , respectively and $\beta = 0$.

the stability characteristics of Couette flows with varying Mach number, but $\bar{\kappa}_{avg} = 3.3$ for all, in Fig. 5.2b. This value of $\bar{\kappa}_{avg}$ is obtained by averaging over the entire range of Mach numbers ($0.1 \leq M \leq 20$). Fig. 5.2b thus separates the effect of varying Mach number, keeping the thermal conductivity constant at an elevated level as compared to the reference flow, whereas Fig. 5.2c has the combined effect of both $\bar{\kappa}_{avg}$ and M varying simultaneously.

It can be observed that enhancement in the thermal conductivity in Fig. 5.2b and 5.2c increases the number of instability loops compared to the reference flow in Fig. 5.2a. This implies that a larger Mach number range of flows are linearly unstable at higher conductivity, and at each Mach number, the unstable disturbances span a wider range of spatial wave numbers. The maximum growth rates observed in Fig. 5.2b and 5.2c are in the range of $\omega_i \simeq 0.05$, which is

higher than those in the reference flow ($\omega_i = 0.004$) by an order in magnitude. Thus, a faster rate of heat diffusion compared to momentum has a strong destabilizing effect on compressible Couette flow, all other factors kept constant.

We notice that the effect of conductivity stratification is much less prominent. The least stable mode in the κ -stratified flow (Fig. 5.2d) has a comparable growth rate as the corresponding κ -averaged flow in Fig. 5.2c. There are, however, additional instability loops in the κ -averaged flow at high Mach numbers ($10 < M < 20$), representing unstable regions spanning over a wider range of streamwise wavenumbers. Therefore, stratification in conductivity is slightly stabilizing.

In the context of incompressible flows, Sameen and Govindarajan [17] study the effect of heat diffusivity on the stability of a channel flow of liquid, with both symmetric and asymmetric heating. They find that the least stable mode is practically unaffected by a change in the heat diffusivity or Prandtl number, as per linear stability theory. This is in contrast to the current stability results in compressible Couette flow, where we notice a strong destabilizing effect of an increase in the magnitude of thermal conductivity. Sorokin [10] reports a similar destabilizing role of thermal conductivity, for a fluid layer over a vertical plane, where relatively small changes in the thermal conductivity can destabilize the flow.

Stability plots in Reynolds number-wavenumber plane

Fig. 5.3 plots the least stable growth rates in the Reynolds number-streamwise wavenumber plane for a two-dimensional disturbance at Mach 10. The stability diagrams for the reference and the κ -stratified flows are presented in Figs. 5.3a and 5.3c respectively, for a reference Prandtl number of 0.72. Fig. 5.3b represents the κ -averaged flow, with an average Prandtl number of 0.21, corresponding to the κ -stratified flow in Fig. 5.3c. Once again, the differences observed in the stability characteristics between the reference flow (Fig. 5.3a) and the κ -averaged flow (Fig. 5.3b) are solely due to an increase in the magnitude of thermal conductivity, with no gradient in $\bar{\kappa}$ across the flow.

An increase in the magnitude of $\bar{\kappa}$ leads to a decrease in the Prandtl number value (from 0.72 to 0.21), which in turn increases the maximum value of the disturbance growth rate by around 10 times – from 0.004 in the reference flow to 0.05 in the κ -averaged flow. The number of instability loops and the area occupied by them in the $Re-\alpha$ plane also increase significantly

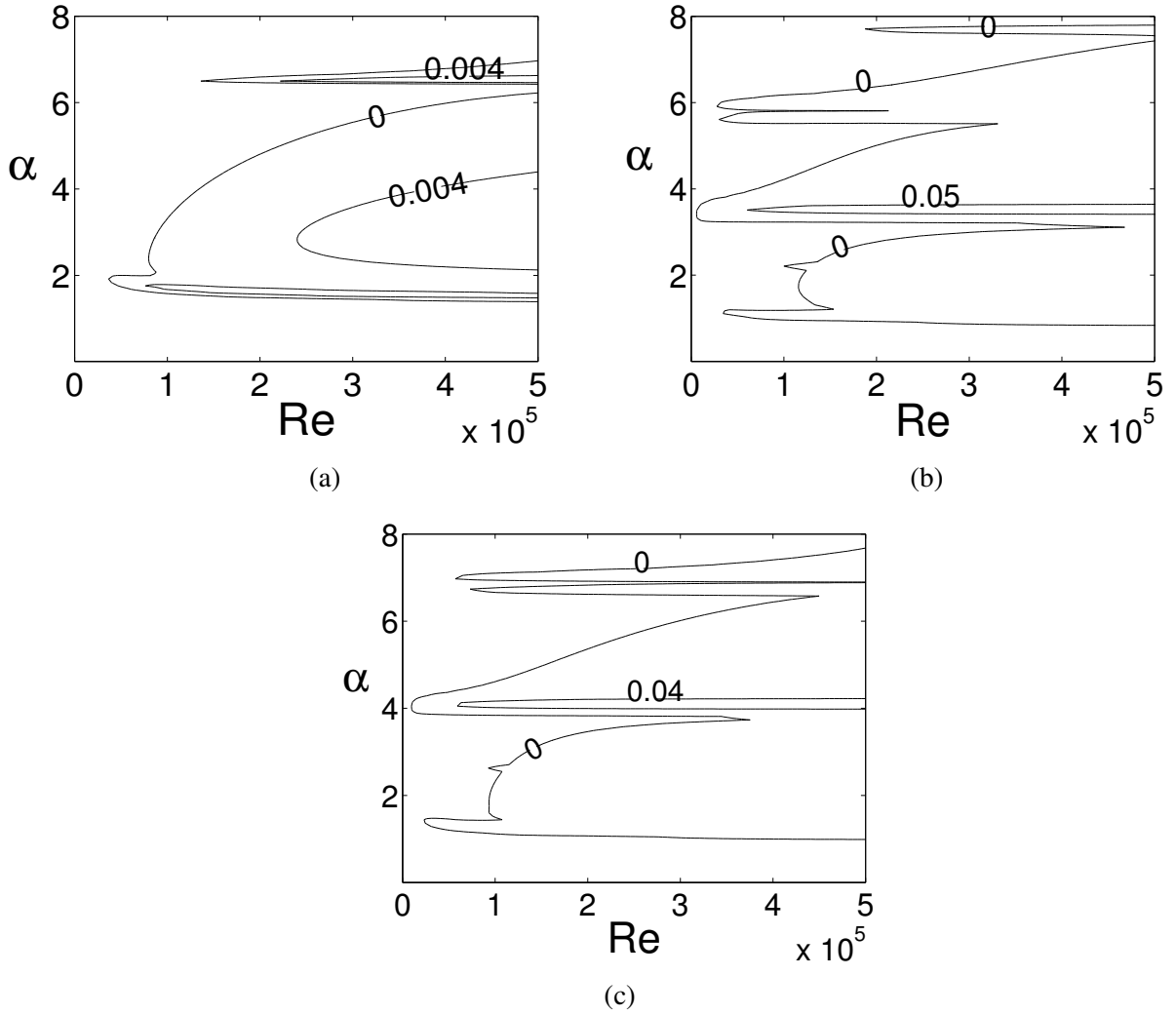


Figure 5.3: Comparison of growth rate contours in the Re - α plane for (a) the reference flow, (b) κ -averaged flow with $Pr_{avg} = 0.21$, and (c) κ -stratified flow. The reference Prandtl number and the Mach number considered are 0.72 and 10 respectively.

for flows with higher thermal conductivity.

Table 5.1 lists the critical Reynolds numbers (Re_{cr}) for the reference, κ -averaged and the κ -stratified flows at different Mach numbers. Two different reference Prandtl numbers (0.72 and 0.2) are considered, and the Pr_{avg} values in the corresponding κ -averaged flows are also tabulated. A large change in the critical Reynolds number is observed due to an increase in the mean thermal conductivity from its reference value of 1 to κ_{avg} . For $M = 10$ and $Pr_{ref} = 0.72$, Re_{cr} decreases from 36437 to 5537. The wavenumber (α_{cr}) corresponding to the critical Reynolds number increases from 1.93 to 3.41, implying a change in the instability loop that contributes to Re_{cr} , from the first to the second loop (in Fig. 5.3a and 5.3b). This is often associated with a change in the dominant instability mode, from first to the second acoustic mode, as discussed in Ref. [71].

The effect of stratification in conductivity can be realized by comparing Figs. 5.3b and 5.3c,

Table 5.1: Comparison of critical Reynolds number and wavenumber at various Mach numbers for three base flows with different conductivities and uniform viscosity. Reference Prandtl numbers considered are 0.72 and 0.2.

Pr	M	Re_{cr}			Pr_{avg}	α_{cr}		
		Reference	κ -avg	κ -strat		Reference	κ -avg	κ -strat
0.72	3	40555	28051	27426	0.46	2.58	2.41	2.52
0.72	5	19527	13290	12252	0.34	2.18	1.79	1.97
0.72	8	27210	6136	10434	0.25	1.98	4.01	4.63
0.72	10	36437	5537	8860	0.21	1.93	3.41	4.03
0.2	3	9704	8311	9241	0.17	2.43	2.41	2.40
0.2	5	10442	6659	7665	0.14	5.22	4.76	4.92
0.2	8	4672	2715	3530	0.10	3.71	3.03	3.27
0.2	10	5232	2474	3831	0.09	3.34	2.51	2.79

which are two flows with the same average value of conductivity, and the difference being the stratification in $\bar{\kappa}$. The κ -averaged flow has slightly higher maximum growth rate (0.05 in Fig. 5.3b as compared to 0.04 in Fig. 5.3c), in addition to an increase in the number of instability loops. However, the most unstable mode comes from the second loop in both the cases. The critical Reynolds numbers for both the cases are relatively close compared to the reference flow as can be seen from the Table 5.1.

Whether conductivity stratification has any significant effect depends on the Mach number and the reference Prandtl number. For example, in the case of $Pr = 0.72$, the κ -averaged flow behaves similar to the κ -stratified flow at $M = 3$ and 5, with a marginal stabilization (see Table 5.1), whereas at higher Mach numbers (8 and 10), the κ -averaged flow is significantly more unstable. On the other hand, for a low reference Prandtl number of 0.2, a stratification in thermal conductivity stabilizes the flow at all Mach numbers. In these cases, κ -averaged flow is the least stable amongst all the three flows.

5.2.2 Transient growth results

Fig. 5.4 presents the maximum temporal energy amplification (G_{max}) as a function of the streamwise and spanwise wavenumber. To investigate the effect of stratification in the thermal conductivity on the short-term energy growth, we compare the κ -stratified flows in Figs. 5.4e and 5.4f with the κ -averaged flows in Figs. 5.4c and 5.4d respectively. The reference flows are presented in Figs. 5.4a and 5.4b. Figs. 5.4a, 5.4c and 5.4e are at a reference Prandtl number of 0.7, while the remaining figures correspond to $Pr_{ref} = 0.2$. We observe that maximum amplification of the transient energy occurs for a streamwise independent disturbance ($\alpha = 0$). This is valid for all the three base flow cases considered in Fig. 5.4, with any Prandtl and Mach number.

We note that an elevated thermal conductivity in the κ -averaged flow has a large effect on the transient energy amplification compared to the stratification in the mean conductivity. The maximum energy amplification increases significantly from the reference to the κ -averaged flow, with minimal changes between the κ -averaged and κ -stratified flows. This trend is valid at both Prandtl numbers, although reducing the reference Prandtl number from 0.7 to 0.2 leads to a higher amplification of energy irrespective of the type of flow considered. The data in Fig. 5.4 corresponds to Mach 5. Similar effects of magnitude and gradient in the conductivity on transient energy amplification are observed at higher Mach numbers, but the energy amplification decreases significantly with an increase in the flow Mach number [65].

The critical Reynolds number corresponding to the κ -averaged flow is lower than the κ -stratified flow at a Mach number and a Prandtl number of 5 and 0.2 respectively (see Table 5.1). The transient growth data also follows the same trend, with the κ -averaged flow having the highest energy amplification (Fig. 5.4d). The trend observed in the transient growth data at the higher Prandtl number ($= 0.7$) is however, opposite to the linear stability results, namely, the κ -stratified flow is the most modally unstable at Mach 5 and $Pr = 0.7$. As opposed to our compressible flow results, for incompressible flows, Sameen and Govindarajan [17] observe a large destabilizing effect of reducing the heat diffusivity or increasing the Prandtl number value on the transient growth of disturbance kinetic energy for a channel flow; though linear stability predicts a negligible effect of varying heat diffusivity.

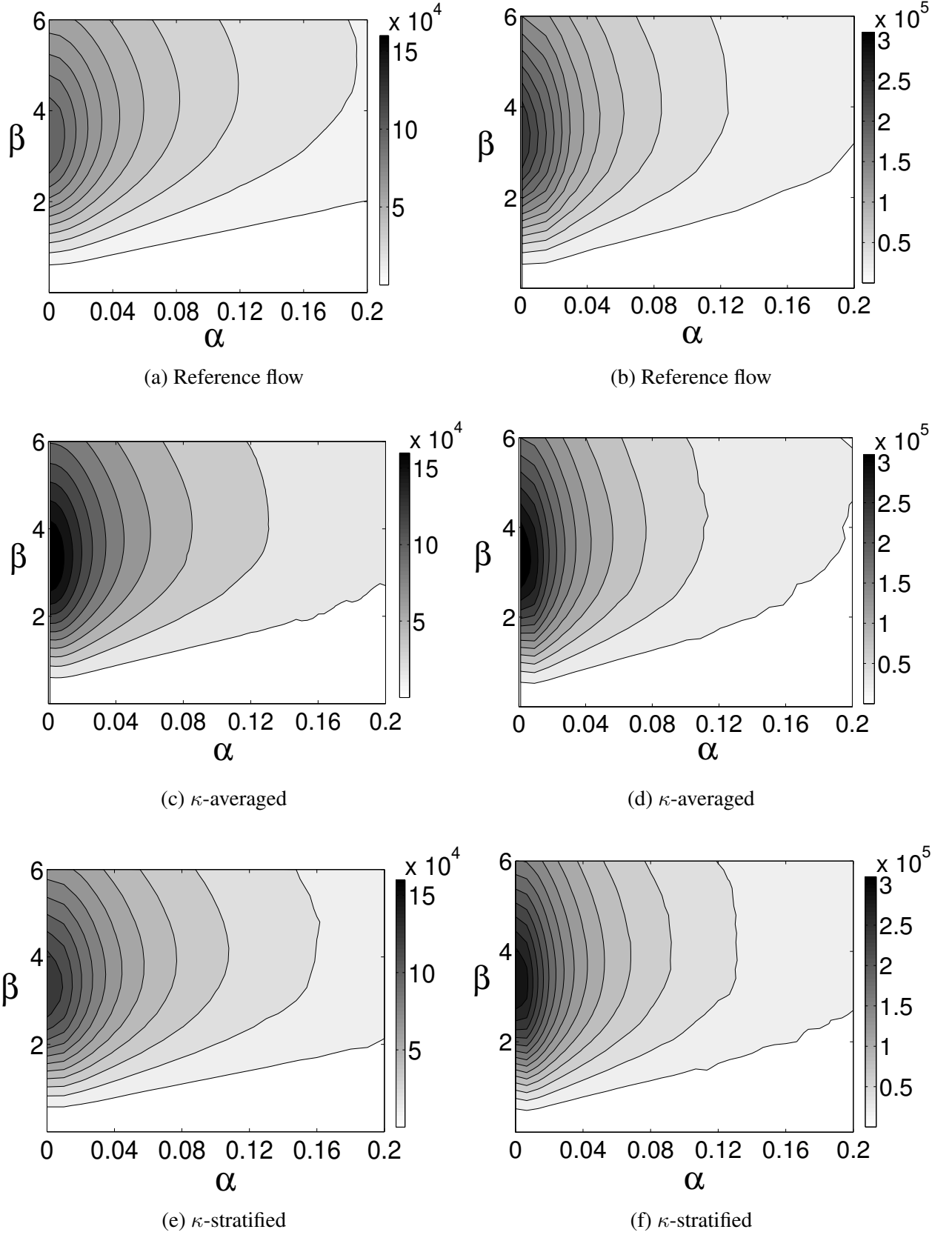


Figure 5.4: Contours of maximum temporal energy amplification G_{max} in the wavenumber plane, showing the effect of increase in the magnitude of conductivity and its stratification at $M = 5$. The reference Pr for (a) and (e) is 0.7, while (b) and (f) are at $Pr = 0.2$. The reference Prandtl number corresponding to the averaged flows (c) and (d) are 0.33 and 0.14 respectively.

Transient energy budget

To get a physical insight into the various energy transfer mechanisms, in this section we present a non-modal budget for the disturbance energy. The rate of change in the total disturbance energy can be expressed as,

$$\frac{\partial E}{\partial t} = -i \int_0^1 \phi^+ \mathbf{M} B \phi dy + c.c. = \frac{\partial P}{\partial t} + \frac{\partial V}{\partial t} + \frac{\partial T}{\partial t} + \frac{\partial S}{\partial t} + \frac{\partial W}{\partial t}. \quad (5.4)$$

The linear stability operator B and coefficient matrix $\mathbf{M}$ are defined in sections 3.2.1 and 3.2.2 respectively, and c.c. denotes complex conjugate. Here, P represents the production of disturbance energy due to mean flow gradients, V stands for the dissipation due to viscous effects, and T represents the thermal diffusion. The work done by the viscous shear is denoted by S , and pressure work is represented by W . These energy constituents are computed as per expressions mentioned in the appendix 2, and the effect of variation in the transport properties on these energy transfer mechanisms will be investigated next.

Fig. 5.5 presents the budget for the perturbation energy corresponding to the optimal disturbance ($\alpha = 0$ and $\beta = 3.6$) in the reference and κ -averaged Couette flows. The energy budget for the κ -stratified flow is almost identical to the κ -averaged case and is not shown in the figure. The parameters considered here are same as in Figs. 5.4b and 5.4d, and the eigenvalues with a growth rate higher than -0.5 are used in the budget computations. The constituent energies in Fig. 5.5 are integrated values across the shear layer. It is found that the production, dissipation and thermal diffusion (to a smaller extent) are the main contributors to the total energy. Their magnitudes increase monotonically in time and reach asymptotic values eventually. The trend is similar to that in Fig. 14a in Ref. [67], which is used to validate our methodology of budget computation. The energy transferred from the mean flow dominates at short-term, leading to a net gain in energy (shown by the solid bold line in Fig. 5.5). The shear work and energy transferred by the pressure forces are significantly less and are not shown in the figure.

Comparing Fig. 5.5a with 5.5b, we notice that the energy transferred from the mean flow to the disturbance is significantly higher in the case of κ -averaged flow compared to the reference case. The viscous forces play a significant role in the κ -averaged flow, and it leads to a higher energy lost by viscous dissipation compared to the reference flow. The energy transferred due to thermal diffusion is found to be negligible compared to the above two terms. The net effect

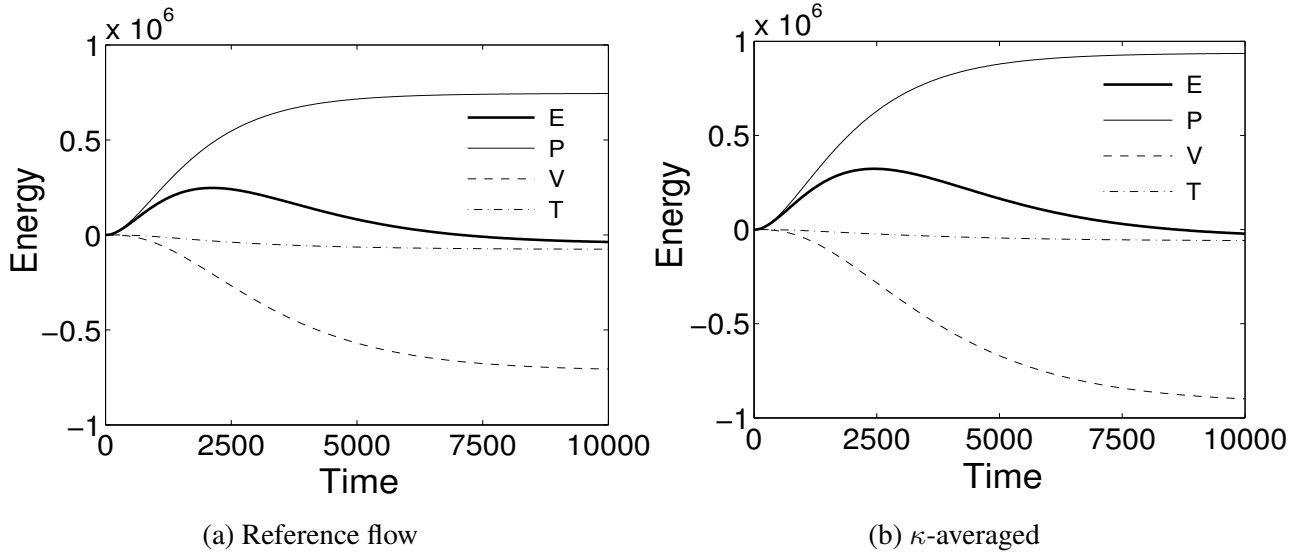


Figure 5.5: Comparison of the energy budget for the reference and the κ -averaged flow corresponding to the maximum transient energy amplification in Fig. 5.4b and 5.4d respectively. Here E, P, V, and T represent total energy, production or energy transferred from the mean flow, viscous dissipation and thermal diffusion respectively, and the variation of each term with time is shown.

of increased production and dissipation in the κ -averaged flow leads to an increase in the peak total energy by a factor of 1.3 compared to the reference flow. Also, the location of the peak energy shifts to a later time, denoting a longer duration of transient energy growth in the flow having a higher thermal conductivity. Stratification of conductivity in the Couette flow is found to have a comparatively smaller effect on the constituent energies, as reported earlier. Therefore an increase in the magnitude of conductivity plays a major role in the redistribution of energy transferred from the mean flow to the disturbances. Hence the energy amplification factors in the κ -averaged flow presented in Fig. 5.4 are higher than the reference flow at comparable wavenumbers.

5.3 Summary

In this chapter, we construct three different model problems to isolate and study the effect of an increase in the magnitude of conductivity and its gradient. These flow configurations are- the reference flow, κ -averaged flow and the κ -stratified flow. By analyzing both the linear stability and transient growth results we find that, a κ -stratified flow behaves much the same as the κ -averaged flow, with the same average value of conductivity and reference viscosity. Also, these two flows are far more unstable, compared to the reference flow at all Mach and reference

Prandtl numbers. In other words, the main effect of conductivity stratification is captured by a net increase in its value. The stability diagrams along with the disturbance energy analysis predict a strong destabilizing role of increasing the thermal conductivity value, leading to larger regions of instability, increased growth rates, and higher temporal energy amplification. A significant reduction in the critical Reynolds number is also observed in the κ -averaged flow compared to the reference flow.

Chapter 6

Viscosity stratified flow

In this chapter, we first discuss the base flow equations governing the viscosity stratified flow. A similar analysis as that of Chapter 5 is carried out to study the effect of variation of viscosity on the stability of compressible Couette flow.

6.1 Base flow

In the case of μ -stratified flow, conductivity is kept constant ($\bar{\kappa} = 1$) at its reference value, and viscosity is varied as a function of temperature using Sutherland's formula (Eq. 6.2). With a constant value of the shear stress ($= \tau$) at the wall, the momentum equation in the streamwise direction (Eq. 3.1) can be written as,

$$\bar{\mu} \frac{d\bar{U}}{dy} = \tau \text{ (constant)}, \quad (6.1)$$

$$\text{where } \bar{\mu} = \bar{T}^{\frac{3}{2}} \frac{(1 + C)}{(\bar{T} + C)}, \text{ and } C = \frac{110.4 \text{ K}}{T_{ref}}. \quad (6.2)$$

For a lower adiabatic wall, using Eq. 6.1, the energy equation (Eq. 3.2) can be integrated such that $\bar{U}$ becomes 1 when $\bar{T} = 1$. This leads to the following equation for temperature as a function of base flow velocity:

$$\frac{\bar{T} - C}{\sqrt{\bar{T}}} = \frac{(\gamma - 1) M^2 Pr}{4} (1 - \bar{U}^2) (1 + C) + (1 - C), \quad (6.3)$$

which can be expressed completely in terms of $\bar{U}$ such that the shear stress can be found iteratively using the shooting method. Eq. 6.4 can be numerically integrated for $\bar{U}$ with the initial

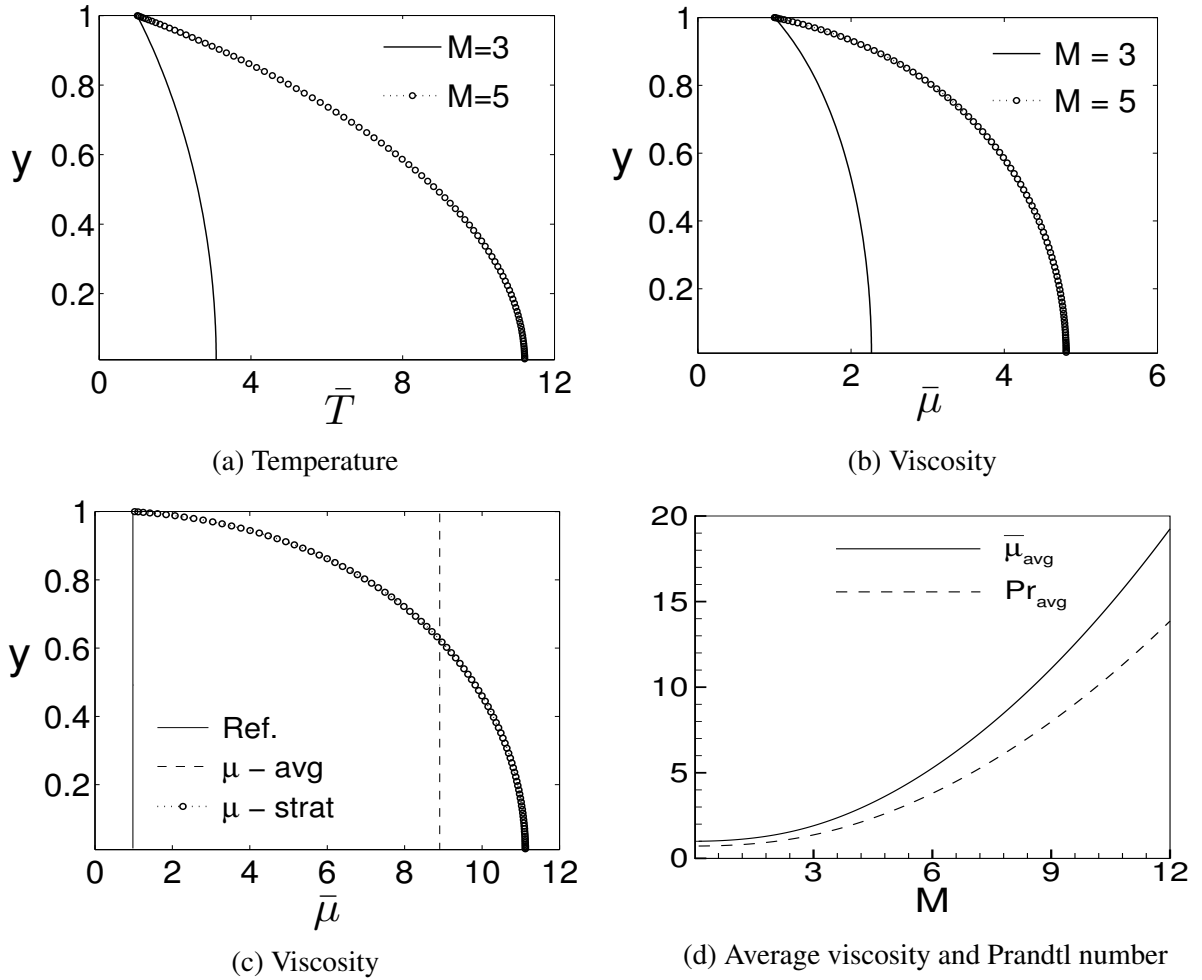


Figure 6.1: The variation of (a) mean temperature and (b) mean viscosity along the wall-normal direction for a μ -stratified flow at two Mach numbers; (c) comparison of viscosity variation in the three base flows at $M = 5$; (d) the average values of viscosity and Prandtl number for a range of Mach numbers. The reference Prandtl number is 0.72.

condition $\bar{U}(0) = 0$, and finding the value of τ for which $\bar{U}(1) = 1$.

$$\frac{d\bar{U}}{dy} = \frac{\tau}{\bar{\mu}(\bar{T})} = \frac{\tau}{\mu(\bar{U})} = f(\bar{U}). \quad (6.4)$$

The base flow obtained for a reference Prandtl number and temperature of 0.72 and 220.8 K respectively is presented in Fig. 6.1. The variation of temperature and viscosity, in this case, are qualitatively the same to those presented in Fig. 5.1a. The temperature levels are, however, much higher in the μ -stratified flow than those in the corresponding κ -stratified cases, due to higher viscous dissipation. The viscosity profiles in the μ -stratified flows can be interpreted as an increase in the average value of viscosity across the entire shear layer plus a viscosity gradient imposed on the flow. Similar to the conductivity cases described in section 5.1, we can define an average value for viscosity as,

$$\bar{\mu}_{avg} = \int_0^1 \bar{\mu} dy, \quad (6.5)$$

and use it to define a μ -averaged model Couette flow, which has a constant viscosity ($\bar{\mu}_{avg}$) and the reference value of conductivity across the shear layer. The viscosity variation in the reference flow, the μ -averaged flow and the μ -stratified flows are shown in Fig. 6.1c for a Mach number of 5. The effect of viscosity stratification is determined by comparing the μ -averaged flow with the μ -stratified flow since both have the same average value of mean fluid viscosity ($\bar{\mu}$). The average fluid viscosity ($\bar{\mu}_{avg}$) and the corresponding average Prandtl number ($Pr_{avg} = \bar{\mu}_{avg} Pr$) are plotted as a function of the Mach number in Fig. 6.1d for a reference Prandtl number of 0.72. It can be observed that the average value of viscosity increases rapidly with Mach number, which leads to a sharp increase in the Pr_{avg} value.

6.2 Stability results

6.2.1 Linear stability results

Stability plots in Reynolds number-wavenumber plane

Fig. 6.2 plots the growth rate contours in the Reynolds number and the streamwise wavenumber plane for a Couette flow, at $M = 10$ and reference $Pr = 0.2$. The stability diagrams for the reference and the stratified viscosity flow are plotted in Figs. 6.2a and 6.2c respectively. The average mean viscosity corresponding the μ -stratified flow at $M = 10$ and $Pr = 0.2$, is taken as the mean viscosity for the μ -averaged flow, presented in Fig. 6.2b.

We notice that for a given Reynolds number, the reference flow is the least stable ($\omega_i \simeq 0.05$ in Fig. 6.2a), with an order of magnitude higher maximum growth rate compared to the other two base flow cases ($\omega_i \simeq 0.005$ in Figs. 6.2b and 6.2c). It also has a larger region of instability in the $Re-\alpha$ plane. The enhancement in the mean viscosity value from 1 to 4.2 and a consequent increase in the Prandtl number (from 0.2 in the reference flow to 0.85 in the μ -averaged flow) makes the μ -averaged flow more stable compared to the reference flow. The reduction in the Reynolds number, caused by an increase in the viscosity value, also contributes to the low growth rates observed in the μ -averaged flow.

The effect of stratification in viscosity is realized by comparing Fig. 6.2b with 6.2c. Both the μ -averaged and μ -stratified flows have similar maximum growth rates, however, the stratification in viscosity shifts the instabilities to a higher Reynolds number. The strong stabilizing

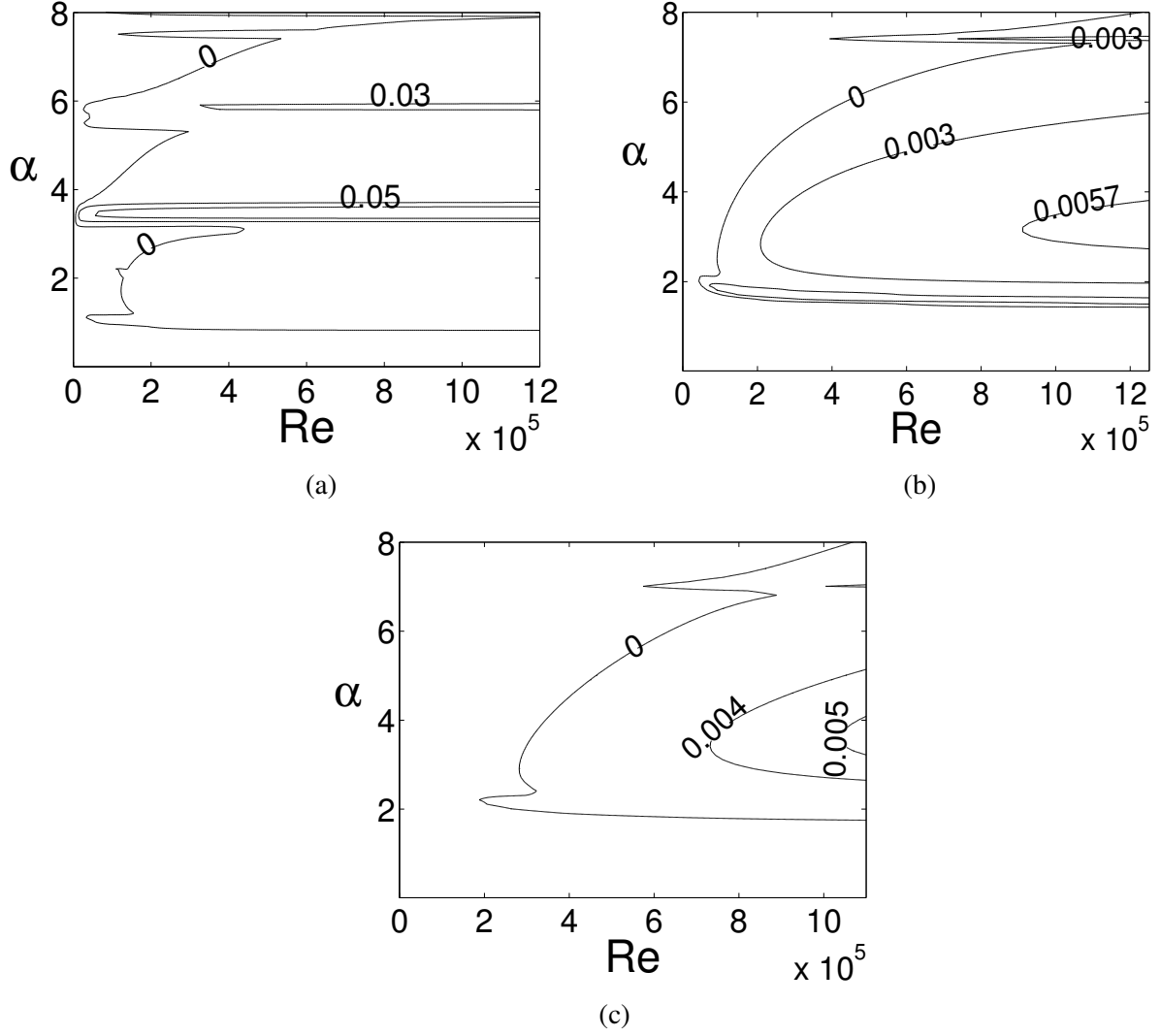


Figure 6.2: Comparison of growth rate contours in the Re - α plane for (a) the reference flow, (b) μ -averaged flow with $Pr_{avg} = 0.85$, and (c) μ -stratified flow. Reference Prandtl number and Mach number considered are 0.2 and 10 respectively.

effect of viscosity gradient can also be noticed from Table 6.1, which lists the critical Reynolds numbers, and the corresponding disturbance wavenumbers, for different values of Mach and reference Prandtl numbers. For example, at $M = 5$ and reference $Pr = 0.5$, the critical Reynolds number for the μ -stratified flow is 283789, which is around five times larger than the corresponding μ -averaged flow. The same trend is also observed at other Mach and Prandtl numbers, with a stronger stabilizing effect of viscosity gradient at higher Mach numbers. Increasing the reference Prandtl number from 0.2 to 0.5, also increases Re_{cr} (see Table 6.1) for all the flow cases considered, and thus leads to a more stable flow at higher Pr . A similar stabilizing effect of viscosity stratification is also reported in Ref. [22] for a compressible Couette flow. However, the work includes the variation of viscosity and conductivity simultaneously, since a constant Prandtl number is used in their formulation.

Table 6.1: The critical Reynolds number (Re_{cr}) and wavenumber (α_{cr}) at various Mach numbers for three different base flows with uniform conductivity, and different values of viscosities. Reference Prandtl numbers considered are 0.5 and 0.2.

Pr	M	Re_{cr}			Pr_{avg}	α_{cr}		
		Reference	μ -avg	μ -strat		Reference	μ -avg	μ -strat
0.5	3	29488	46014	147874	0.8	2.44	2.63	2.77
0.5	5	15136	53397	283789	1.46	1.98	2.63	2.79
0.5	7	18860	233130	2416502	2.52	1.79	3.00	3.17
0.2	3	9704	12148	20784	0.24	2.43	2.45	2.26
0.2	5	10442	13267	25484	0.34	5.22	1.79	1.92
0.2	10	5232	41876	187900	0.85	3.34	2.05	2.21

In contrast to our compressible results, where we observe a strong stabilizing role of an increase in the magnitude of viscosity, the linear stability results for incompressible flows predicts that decreasing the viscosity towards the wall is stabilizing. Also, if there exists any temperature difference in the flow, the critical Reynolds number increases for low-speed flows. Amongst various studies, this is recently shown in Ref. [17], for a channel flow of liquids, maintained between two walls of different temperatures.

6.2.2 Transient growth results

To study the effect of variation in mean viscosity ($\bar{\mu}$) and its stratification on the short-term dynamics of the flow, we plot the contours of maximum temporal energy amplification on the wavenumber plane in Fig. 6.3. The Mach number, Reynolds number and reference Prandtl number considered are 5, 10^5 and 0.2 respectively. Figs. 6.3a, 6.3b and 6.3c represent the reference, μ -averaged, and μ -stratified flows respectively. The μ -averaged flow has a higher mean viscosity, which leads to a higher Prandtl number ($= 0.34$) in the flow, compared to the reference case ($Pr = 0.2$).

We observe that the initial energy of the disturbance in the reference flow amplifies significantly more compared to the other two flows (see Fig. 6.3). Increase in viscosity from 1 in the reference flow to 1.7 in the μ -averaged flow, leads to a lower energy amplification in the

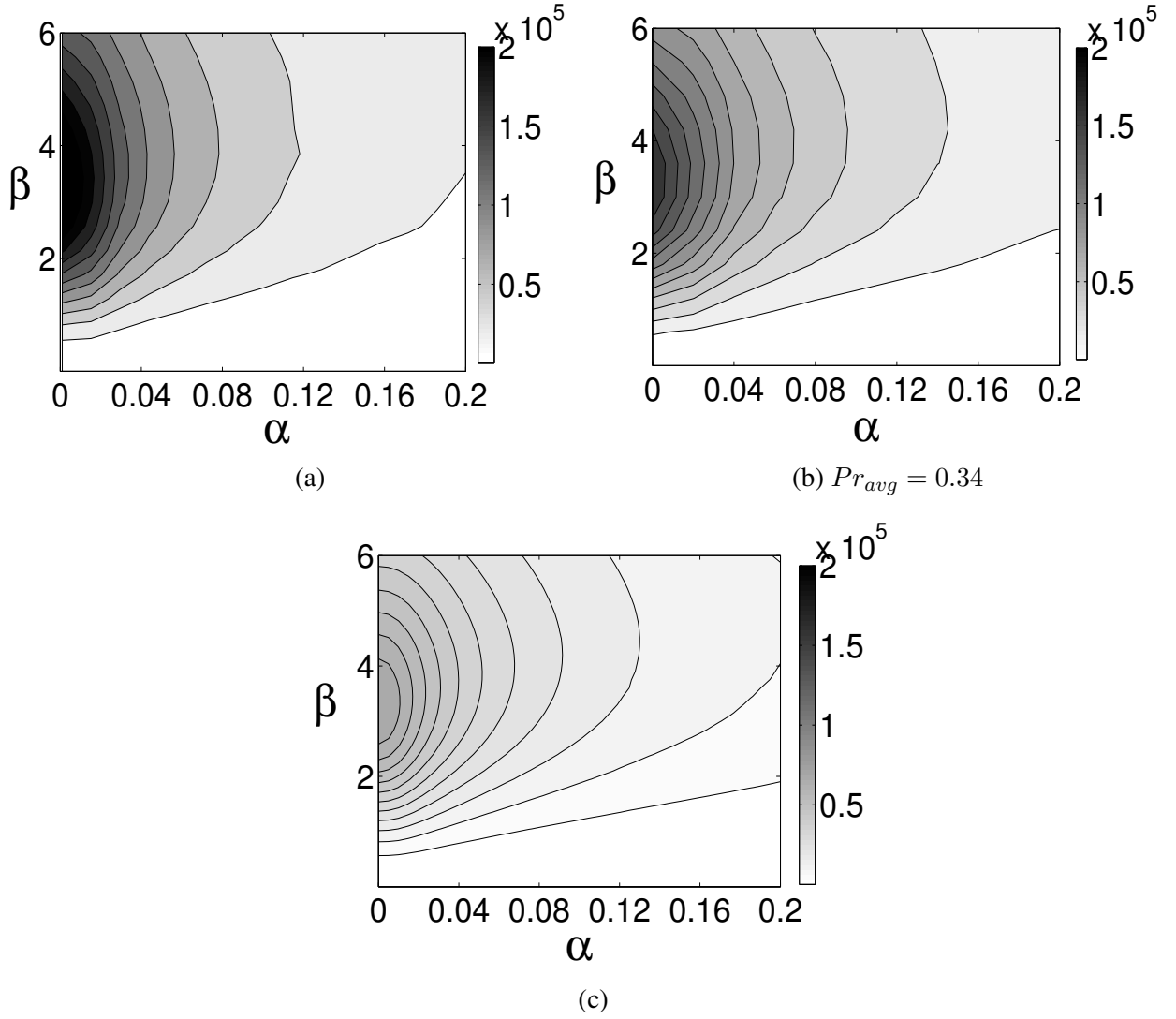


Figure 6.3: Contours of maximum temporal energy amplification G_{max} in the wavenumber plane, showing the effect of stratification in viscosity. Figures (a), (b) and (c) represent the reference, μ -averaged and μ -stratified flows, respectively at $M = 5$ and $Re = 10^5$. The reference Prandtl number is 0.2.

case of averaged flow. The effect of viscosity stratification on transient energy amplification is obtained by comparing Fig. 6.3b with 6.3c, since both the flows have the same average value of viscosity. We notice that the stratification in viscosity reduces the energy amplification significantly, making it the most stable among these flows. It can be noted that the maximum energy amplification at all Mach and Prandtl numbers occurs for a spanwise disturbance ($\alpha = 0$), similar to the κ -varying flows discussed in section 5.2.2. Ref. [22] reports a similar effect of viscosity stratification on the maximum temporal growth of perturbation energy, however, this study includes the effect of thermal conductivity as well.

To examine the effect of viscosity stratification on the transient energy growth, Chikkadi et al. [16] study two types of channel flow. The first one with shear-thinning fluids, and another case with two miscible fluids of the same density, but different viscosities. Sameen et al. [17]

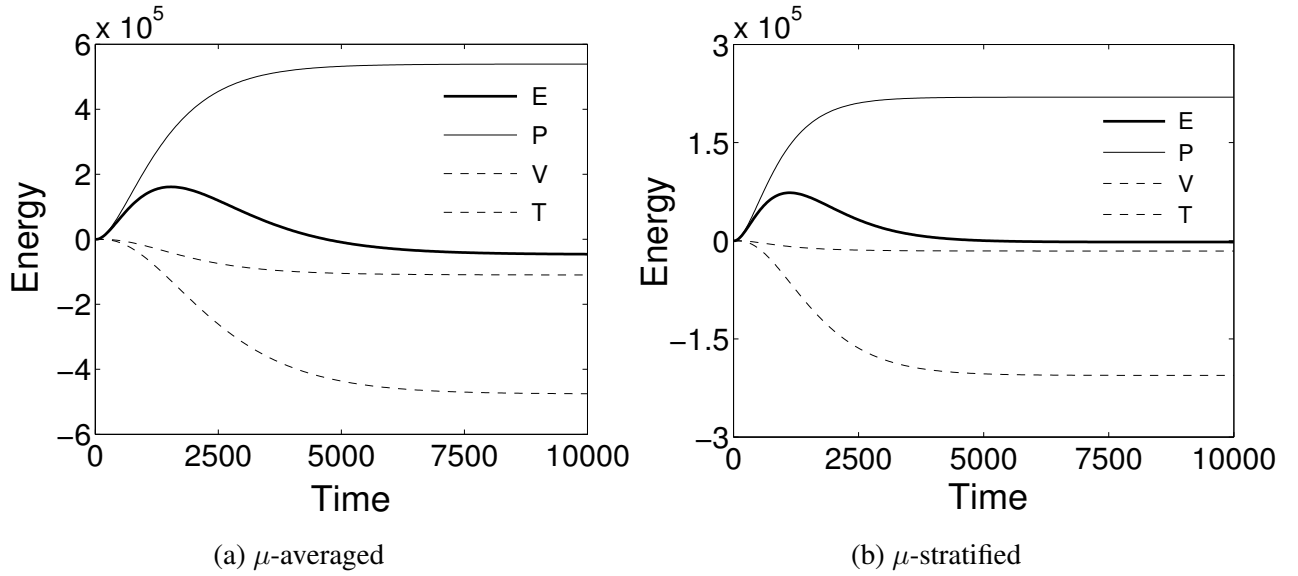


Figure 6.4: Comparison of the energy budget for the total perturbation energy for the μ -averaged and the μ -stratified flow corresponding to the maximum transient energy amplification in Fig. 6.3b and 6.3c respectively.

consider a symmetrically and an asymmetrically heated channel flow, and neglect the variation of Prandtl number and buoyancy, for isolating only the effect of stratification in viscosity. Both Ref. [16] and [17] report that the amplification of disturbance kinetic energy and streamwise vortices are practically unaffected by viscosity stratification.

Transient energy budget

In this section, we discuss the effect of stratification in viscosity on different constituents of perturbation energy. We plot the energy budgets for the μ -averaged and the μ -stratified flow for parameters identical to those in Figs. 6.3b and 6.3c respectively. The data in Fig. 6.4 correspond to the optimal disturbances ($\alpha = 0$ and $\beta = 3.6$) that lead to the maximum transient energy amplification in each case. It is found that the peak total energy in a μ -stratified flow is around two times lower than the μ -averaged case. This can be attributed to a decrease in the energy transfer from the mean flow to the disturbance, which is around 2.5 times less compared to the averaged flow. Stratification in viscosity also leads to a significant reduction in the energy lost by viscous dissipation and thermal diffusion. The decrease in the magnitude of the total energy and its constituents leads to lower gain factors in the μ -stratified flow compared to the averaged flow (see Figs. 6.3b and 6.3c).

6.3 Summary

In this chapter, we study the effect of an increase in the magnitude of viscosity and its gradient on the stability of compressible Couette flow for a range of parameters. We find that the stratification in viscosity has a dramatic stabilizing effect, compared to an increase in the mean viscosity value. This result is valid at a finite time as well as in the asymptotic limit, irrespective of Mach number, Reynolds number and the Prandtl number of the flow. An analysis of the transient perturbation energy budget reveals that the presence of the gradient in viscosity reduces the energy intake from the mean flow significantly, which leads to a much lower energy amplification in the μ -stratified flow.

Chapter 7

Viscosity & conductivity stratified flow

In this chapter, we analyze the combined effect of viscosity and conductivity stratification on the short-term growth of disturbance energy as well as in the asymptotic limit. At first, equations describing the μ - κ -stratified flow is derived, and then a stability analysis corresponding to this base flow is carried out. An energy budget for the total disturbance energy is also presented to examine the role of combined variation of viscosity and conductivity on various physical mechanisms associated with energy transfer.

7.1 Base flow

In the case of a μ - κ -stratified flow, thermal conductivity and viscosity vary independently with temperature, as per Eqs. 5.1 and 6.2, respectively. From Eqs. 3.1 and 3.2, we obtain

$$\frac{d^2 \bar{T}}{dy^2} + \frac{1}{\bar{\kappa}} \frac{d\bar{\kappa}}{d\bar{T}} \left(\frac{d\bar{T}}{dy} \right)^2 + \frac{(\gamma - 1) M^2 Pr \tau^2}{\bar{\mu} \bar{\kappa}} = 0, \quad (7.1)$$

which is solved using the 4th order Runge Kutta method, where the shear stress τ and the lower wall temperature are found iteratively such that $\bar{U}(1) = 1$ and $\bar{T}(1) = 1$. Fig. 7.1 presents the base flow velocity and temperature profiles for a μ - κ -stratified flow at $M = 2$ and 5. The reference temperature and the Prandtl number considered are 220.8 K and 0.72 respectively. Comparing Fig. 7.1b with Figs. 6.1a and 5.1a, we can infer that temperature in a μ - κ -stratified flow is intermediate between the μ -stratified and κ -stratified flows, with the same reference values of Mach and Prandtl numbers.

To determine the effect of simultaneous gradients in viscosity and conductivity, an average

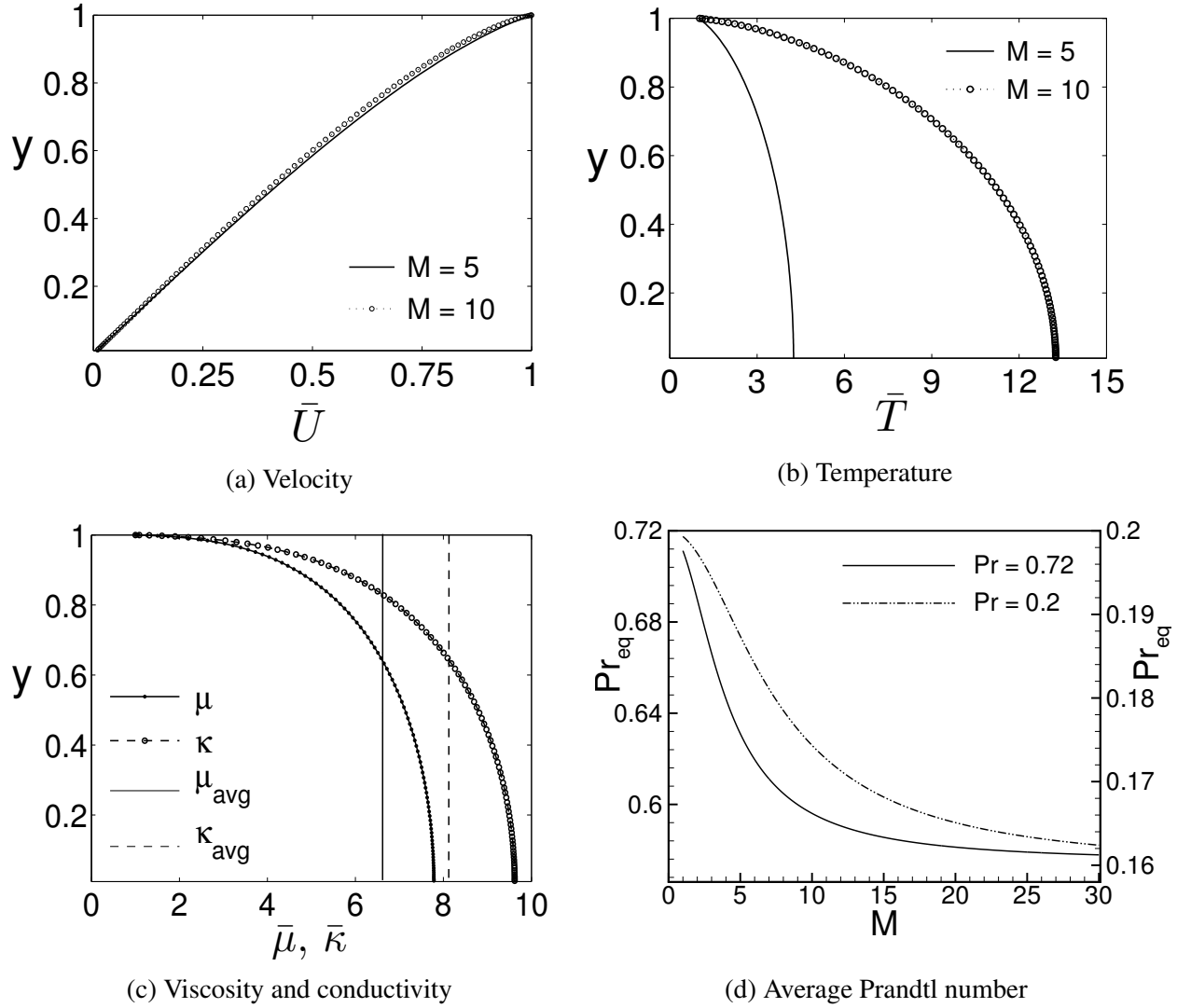


Figure 7.1: The variation of (a) mean velocity and (b) mean temperature along the wall-normal direction for a μ - κ -stratified flow at two Mach numbers; (c) comparison of viscosity and conductivity variation in the three model base flows at $M = 5$; (d) comparison of the average Prandtl number for a range of Mach numbers at a reference Prandtl number of 0.72 and 0.2.

flow is constructed with constant values of $\bar{\mu}$ and $\bar{\kappa}$, obtained from the equivalent μ - κ -stratified flow, by taking the average of viscosity and conductivity (using Eqs. 6.5 and 5.3). The effect of an increase in the magnitude of both $\bar{\mu}$ and $\bar{\kappa}$ on flow stability can be realized by comparing the μ - κ -averaged flow with the reference flow, having $\bar{\mu}$ and $\bar{\kappa}$ equal to 1. The variations in viscosity and conductivity along the wall-normal direction are shown for a μ - κ -stratified flow, at $M = 5$ and $Pr = 0.72$, along with their corresponding average values in Fig. 7.1c. It can be noted that for the current transport model considered, thermal conductivity has a larger variation with temperature compared to viscosity, which results in a lower value of the average Prandtl number ($Pr_{avg} = Pr \bar{\mu}_{avg}/\bar{\kappa}_{avg}$) at higher Mach numbers (see Fig. 7.1d).

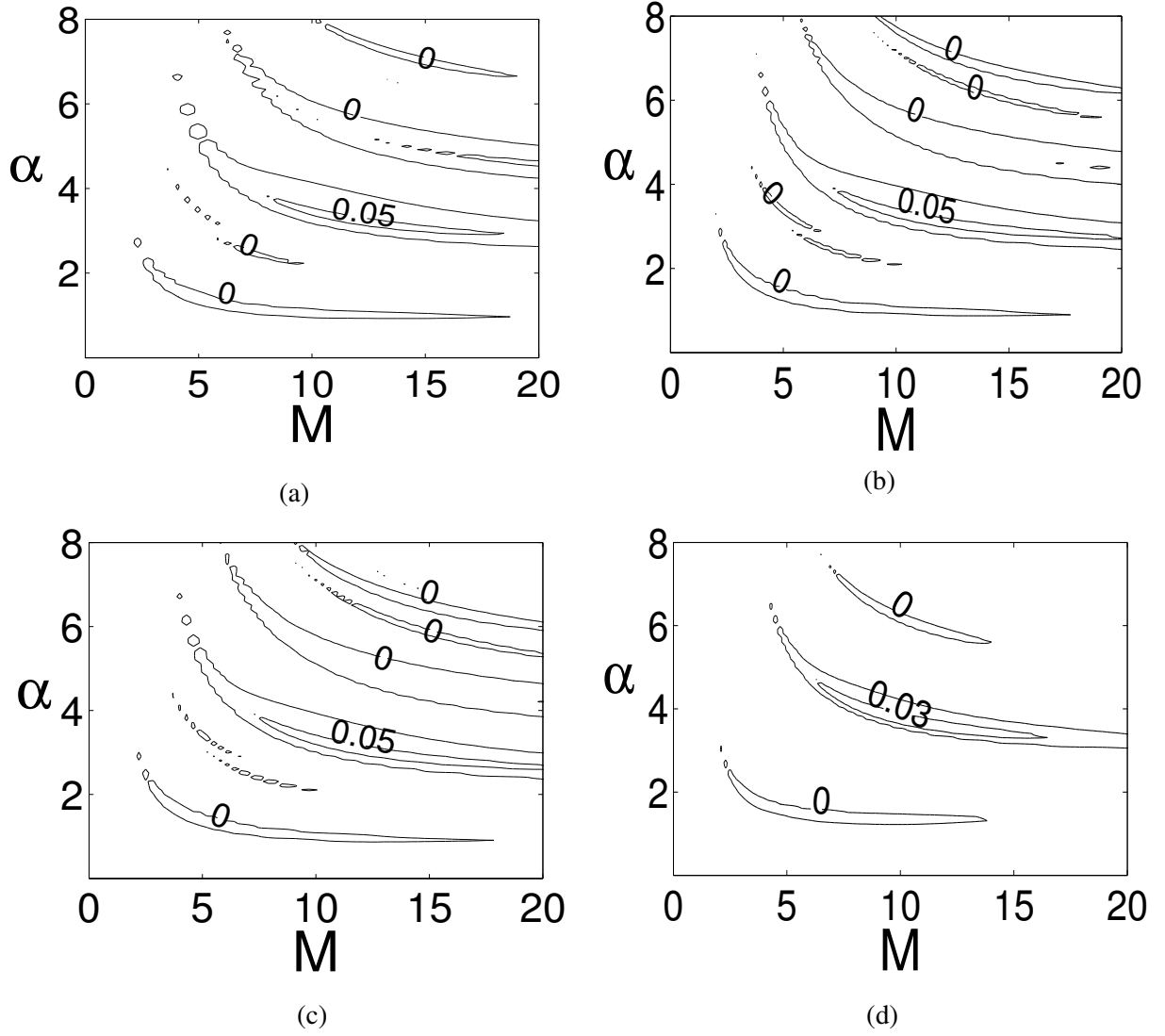


Figure 7.2: Stability diagrams in the Mach-streamwise wavenumber plane for (a) the reference flow, (b) μ - κ -averaged flow with $\bar{\kappa}_{avg} = 3$ and $\bar{\mu}_{avg} = 2.6$, (c) μ - κ -averaged flow with $\bar{\kappa}_{avg}$ and $\bar{\mu}_{avg}$ varying with Mach number, and (d) μ - κ -stratified flow. The reference Prandtl number is 0.2, $\beta = 0$ and $Re = 10^5$.

7.2 Stability results

7.2.1 Linear stability results

Stability plots in Mach-wavenumber plane

Fig. 7.2a presents the reference flow ($\bar{\mu} = \bar{\kappa} = 1$) with the same reference Prandtl number as that of the μ - κ -stratified flow (Fig. 7.2d), in which viscosity and conductivity are varying independently as functions of temperature. Fig. 7.2b represents the μ - κ -averaged flow, with constant values of fluid viscosity ($= 2.6$) and thermal conductivity ($= 3$). These are representative values averaged over a range of Mach numbers from 0.1 to 20 for an equivalent μ - κ -stratified flow (at reference $Pr = 0.2$). Fig. 7.2c is also an averaged flow, but in this case, the values of viscosity

and conductivity at each Mach number are different. These are averaged values at each Mach number corresponding to the μ - κ -stratified flow of Fig. 7.2d.

The effect of an increase in the magnitude of mean viscosity and conductivity is obtained by comparing the stability characteristics of Fig. 7.2a with 7.2b. We notice that both the flows have the same maximum disturbance growth rate ($= 0.05$), but an increase in the transport property values makes the averaged flow slightly more unstable compared to the reference case. The signature of the presence of stratification in both viscosity and conductivity on flow stability can be realized by comparing Fig. 7.2c with 7.2d since both the flows have the same average values of conductivity and viscosity at each Mach number. We notice that most of the regions in the M - α plane are stable in the case of μ - κ -stratified flow (Fig. 7.2d), in contrast to the μ - κ -averaged flow (Fig. 7.2c). In addition, there is a reduction in the maximum growth rate at comparable Mach and disturbance wavenumbers. This implies that simultaneous stratification of $\bar{\mu}$ and $\bar{\kappa}$ has a stabilizing role. This is in line with our previous observation that viscosity stratification is strongly stabilizing, while variation in thermal conductivity has a minor effect on flow stability. In addition, conductivity stratification has a mixed effect of stabilizing or destabilizing depending on the Mach number and Prandtl number of the flow. Thus, the stability of Couette flows with simultaneous variations in viscosity and thermal conductivity is found to be dominated by the viscosity stratification effects, and this is further elucidated by the Re - α plots presented next.

Stability plots in Reynolds number-wavenumber plane

Fig. 7.3 presents the least stable growth rates in the Reynolds number versus wavenumber plane for three model Couette flows with different values of viscosity and conductivity. As earlier, the reference Prandtl number and Mach number considered are 0.2 and 10 respectively, and disturbances are assumed to be two-dimensional. The average Prandtl number for the μ - κ -stratified flow at $M = 10$ and $Pr = 0.2$ is around 0.17, which is the reference Prandtl number for the μ - κ -averaged flow.

Once again, the stability diagrams of the reference and μ - κ -averaged flow are similar in terms of the regions of instability and their growth rates (compare Figs. 7.3a and 7.3b). The critical Reynolds numbers listed in Table 7.1 at different Mach numbers show that the μ - κ -averaged flow is marginally unstable compared to the reference flow for all Mach numbers and

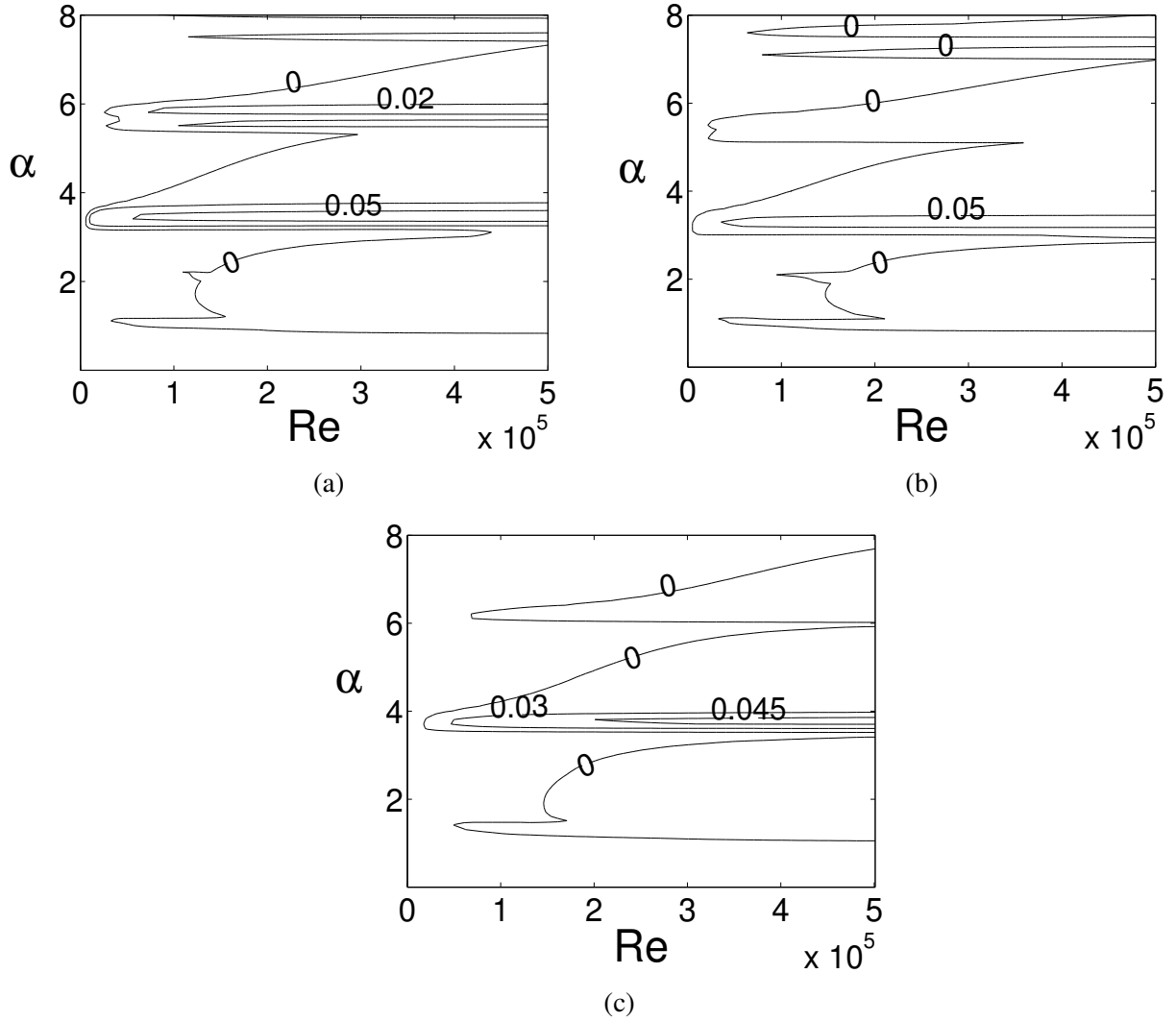


Figure 7.3: Comparison of stability diagrams in the Reynolds number-streamwise wavenumber plane for (a) the reference flow, (b) μ - κ -averaged flow with $Pr_{avg} \approx 0.17$, (c) μ - κ -stratified flow. The reference Prandtl number considered is $Pr = 0.2$ and $M = 10$.

at Prandtl numbers 0.72 and 0.2. The comparison of the critical wavenumbers predicts that the same instability loop (second loop in Fig. 7.3) contributes to the critical Reynolds number for both the flows.

Comparison of Fig. 7.3b with 7.3c shows that the stratification in both viscosity and conductivity decreases the maximum growth rate slightly, but the instability regions along the disturbance wavenumber axis decrease considerably. This stabilizing effect of $\bar{\mu}$ and $\bar{\kappa}$ stratification is more prominent at high reference Prandtl numbers, as can be observed from Table 7.1. At Mach 5, for example, stratification in viscosity and conductivity increases the Re_{cr} by a factor of two for reference $Pr = 0.2$. The critical Reynolds number is enhanced by a factor of more than three for the same flow at $Pr = 0.72$. We notice that for all Mach and Prandtl number combinations, the μ - κ -stratified flow has the maximum critical Reynolds number, while μ - κ -

Table 7.1: Comparison of the critical Reynolds number and wavenumber at various Mach numbers for three different model Couette flows, with varying viscosity and conductivity. Reference Prandtl numbers considered are 0.72 and 0.2.

Pr	M	Re_{cr}			Pr_{avg}	α_{cr}		
		Reference	μ - κ -avg	μ - κ -strat		Reference	μ - κ -avg	μ - κ -strat
0.72	3	40555	37323	111026	0.67	2.58	2.55	2.85
0.72	5	19527	17497	54115	0.63	2.18	2.10	2.57
0.72	10	36437	32333	131551	0.59	1.93	1.79	2.45
0.2	3	9704	9444	18316	0.194	2.43	2.43	2.24
0.2	5	10442	9488	18466	0.187	5.22	5.12	1.76
0.2	10	5232	4540	15840	0.174	3.34	3.17	3.70

averaged flow has the least Re_{cr} value. Therefore, in terms of the linear stability results, we can infer that the presence of stratification in viscosity and conductivity can delay flow transition considerably.

7.2.2 Transient growth results

To investigate the effect of combined variations in viscosity and conductivity on the transient energy growth, we plot the contours of maximum temporal energy amplification in Fig. 7.4. The first three sets of figures (7.4a, 7.4c and 7.4e) are at a Mach number of 5, whereas the next three are shown for a higher Mach number of 10. The Reynolds number and reference Prandtl number considered are 10^5 and 0.2 respectively. A higher increase in conductivity compared to viscosity, for a given temperature profile, leads to a lower Prandtl number in the averaged flows, in comparison with the corresponding reference flows. The μ - κ -averaged flows with a Prandtl number of around 0.19 at $M = 5$ and 0.17 at $M = 10$ are shown in Figs. 7.4c and 7.4d respectively. Similar to the previous transient growth results discussed in sections 5.2.2 and 6.2.2, the maximum energy amplification for the reference and the μ - κ -averaged flow occurs for a disturbance with $\alpha = 0$. However, in the case of μ - κ -stratified flow, transient energy reaches a maximum value at finite streamwise and spanwise wavenumbers (see Figs. 7.4e and 7.4f).

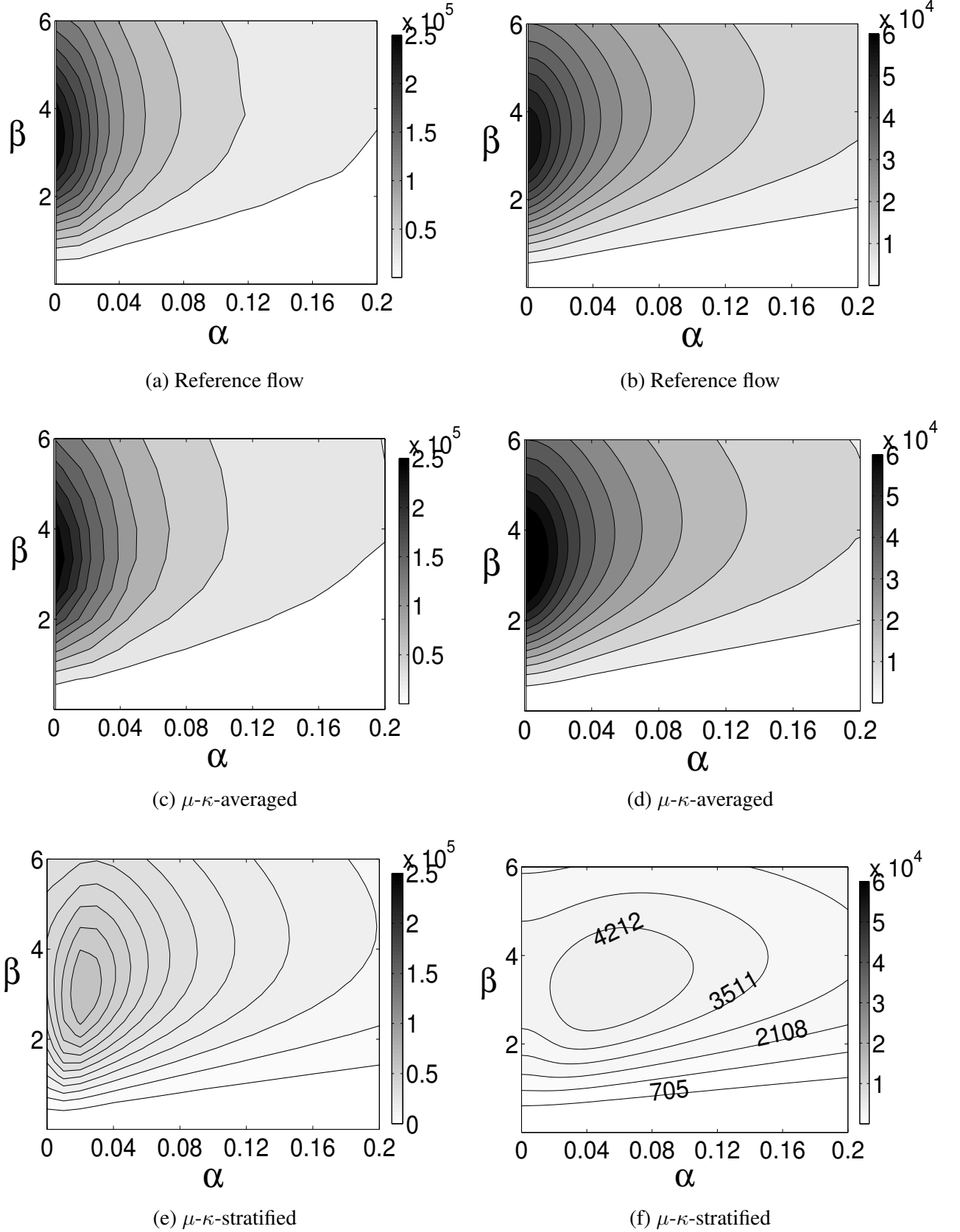


Figure 7.4: Contours of maximum temporal energy amplification in the wavenumber plane showing the effect of stratification in viscosity and conductivity. Figures (a), (c) and (e) are at a Mach number of 5, while (b), (d) and (f) are plotted at $M = 10$. Here, Reynolds number and the reference Prandtl number are 10^5 and 0.2 respectively. The reference Prandtl numbers corresponding to the averaged flows (c) and (d) are 0.187 and 0.174 respectively.

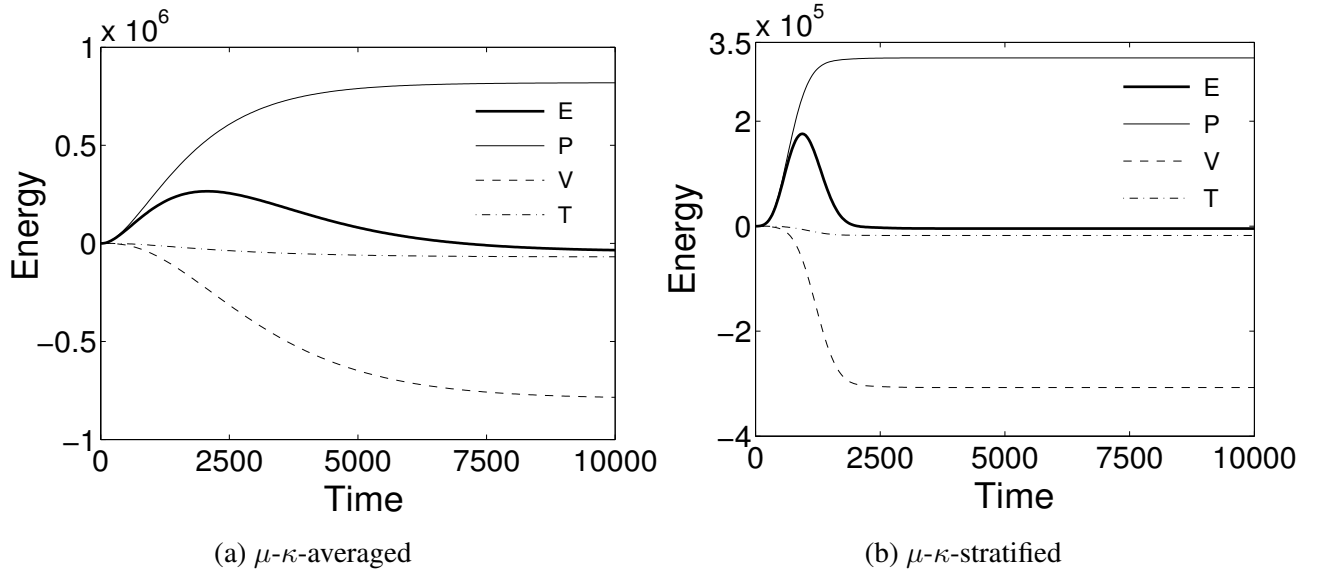


Figure 7.5: Comparison of energy budget for the total perturbation energy for the μ - κ -averaged and the μ - κ -stratified flow corresponding to the maximum transient energy amplification in Figs. 7.4c and 7.4e respectively.

The transient growth analysis exhibits similar results of an increase in $\bar{\mu}$ and $\bar{\kappa}$, and their stratification, as predicted by the linear stability theory. The energy amplification in the μ - κ -averaged flow (Fig. 7.4c) is comparable to that of the reference flow (Fig. 7.4a). On the other hand, a simultaneous variation of viscosity and conductivity (Fig. 7.4e) shows significantly lower transient energy amplification than the μ - κ -averaged flow (Fig. 7.4c), with uniform transport properties maintained at the respective averaged values. Increasing the Mach number from 5 to 10 leads to a considerable decrease in the disturbance energy growth for all the three base flow cases considered, however, the relative magnitudes between the reference, μ - κ -averaged, and μ - κ -stratified cases are identical to that at the lower Mach number.

It can thus be noted that, in a μ - κ -stratified flow, viscosity stratification plays a dominant role compared to the stratification in thermal conductivity. The destabilizing effect of an increase in conductivity is overcome by the presence of stratification in viscosity, due to which μ - κ -stratified flow is far more stable compared to a flow with uniform transport properties.

Transient energy budget

In this section, we analyze the combined effect of viscosity and conductivity stratification on various energy transfer mechanisms by comparing the energy budgets for a μ - κ -stratified flow with the corresponding μ - κ -averaged flow. Figs. 7.5a and 7.5b plot the constituent energies corresponding to the base flow parameters in Figs. 7.4c and 7.4e respectively, for the wavenumber

values which leads to the maximum transient energy amplification in each case. Comparison of Fig. 7.5b with 7.5a shows that in the case of μ - κ -stratified flow, all the energy transfer mechanisms have a sharp variation at a small time, and rapidly approach their respective asymptotic values. The combined effect of viscous forces and thermal diffusion dissipates all the energy transferred from the mean flow, leading to a negligible value of the total energy beyond $t = 2000$. In the case of μ - κ -averaged flow we notice a gradual increase in the different constituents of total perturbation energy, and the peak in the total energy appears at a later time compared to the stratified case. Further, stratification in viscosity and conductivity also leads to a significant reduction (~ 2.6 times) in the energy intake from the mean flow, which causes less energy available to redistribute and dissipate by various physical energy transfer mechanisms. Overall, the stratified flow has significantly lower total energy content in the disturbance, as compared to the corresponding μ - κ -averaged flow. This is in line with our earlier finding that viscosity stratification dominates over conductivity variation, which leads to a smaller transient energy amplification in the case of μ - κ -stratified flow compared to the μ - κ -averaged flow, as seen from Fig. 7.4.

7.3 Summary

In this chapter, we study the effect of an increase in the magnitude of both viscosity and conductivity and their gradient on the stability of a compressible Couette flow using both linear stability theory and transient energy growth. We find that a simultaneous increase in the viscosity and conductivity value slightly destabilizes the flow. However, the presence of their gradients strongly stabilizes the compressible Couette flow. In a μ - κ -stratified flow, viscosity stratification plays a dominant role compared to the stratification in thermal conductivity. It is also observed that the optimal perturbation in the case of μ - κ -stratified flow is not stream-wise independent as that of the reference and the averaged flows. The transient energy budget presented for the total disturbance energy shows that the stratification in viscosity and conductivity leads to a significant reduction in the energy intake from the mean flow, which causes less energy available to redistribute and dissipate by various physical energy transfer mechanisms. We also observe that the energy amplification occurs over a smaller time scale in the case of μ - κ -stratified Couette flow, and all the energy transfer mechanisms decay rapidly.

Chapter 8

Mechanisms leading to transient growth

In this chapter, we study the various physical mechanisms responsible for the non-modal energy growth. The focus is to examine the set of initial conditions that lead to the maximum transient energy amplification and also to examine the structure of the optimal disturbance.

By studying an unbounded constant shear flow of an incompressible fluid, Farrell and Ioannou [72] find that the main factor deciding the physical mechanism leading to transient energy growth is the ratio of spanwise to the streamwise wavenumber ($r = \beta/\alpha$). The Orr mechanism is present when $r = 0$, (i.e., $\beta = 0$) and the lift-up mechanism is observed in the limit of r tending to infinity ($\alpha = 0$). For the intermediate values of the wavenumber ratio, both the Orr and lift-up mechanisms are at work. A brief description of these mechanisms is given below.

8.1 Lift-up mechanism

In the previous chapters, we observe that streamwise independent disturbances lead to the maximum transient energy growth in the case of reference, averaged, κ -stratified and μ -stratified flows. The lift-up mechanism is responsible for the energy amplification in these flows. The energy growth occurs due to the displacement of low-speed fluid near the wall to the high-speed region by the action of wall-normal velocity, keeping the streamwise momentum conserved. Integrating the linearized perturbation equations for an inviscid, compressible and streamwise independent disturbance we obtain,

$$\hat{u} = \hat{u}_0 - \hat{v}_0 \frac{d\bar{U}}{dy} t, \quad (8.1)$$

$$\hat{\rho} = \hat{\rho}_0 - \hat{v}_0 \frac{d\bar{\rho}}{dy} t, \quad (8.2)$$

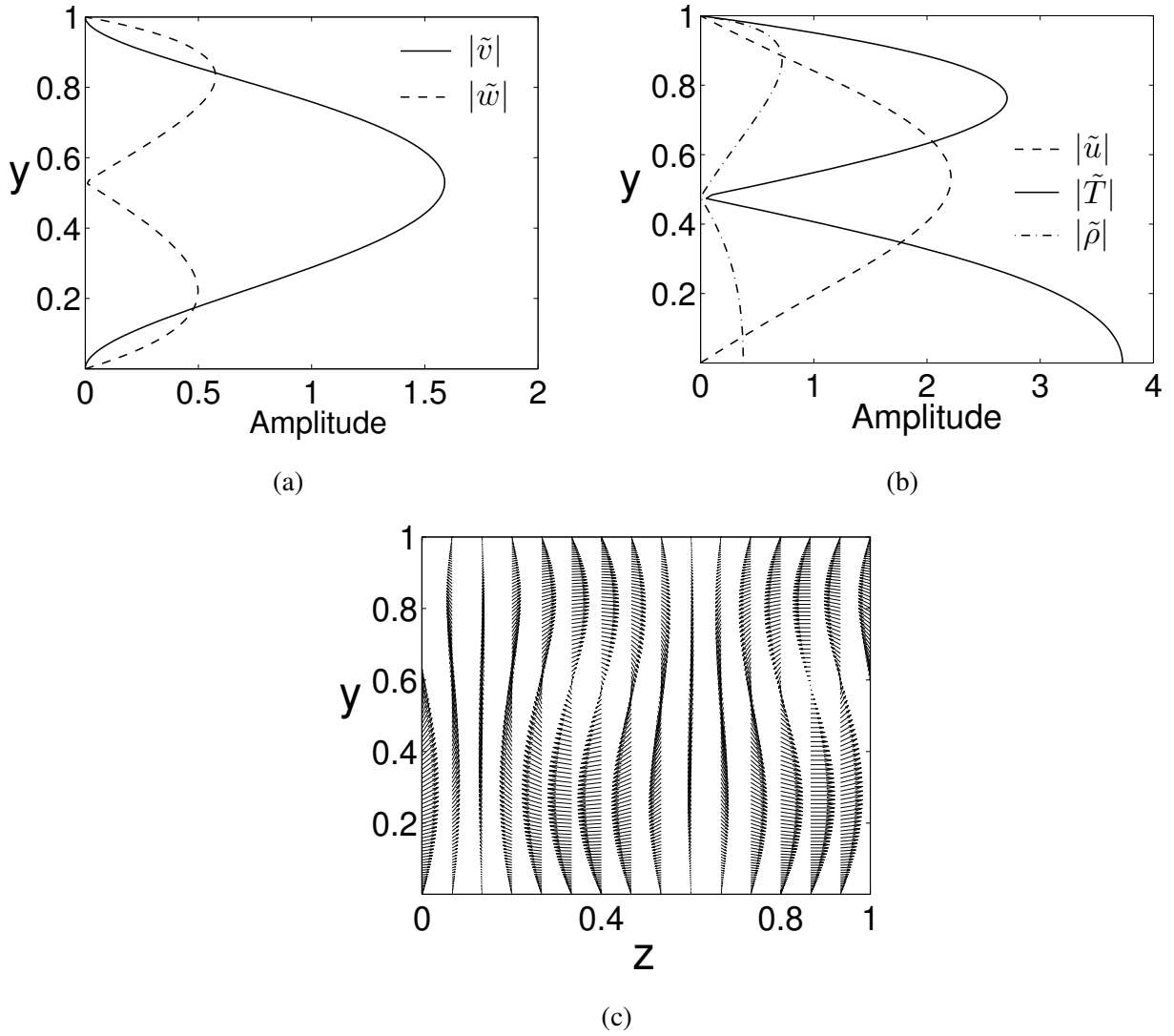


Figure 8.1: Optimal disturbance for a κ -stratified Couette flow at $M = 10$, $Re = 10^5$, $\alpha = 0$, $\beta = 3.36$, and reference $Pr = 0.2$. Figures (a) and (b) show the amplitude of the optimal disturbance before and after amplification, respectively. The flow pattern of the optimal disturbance in $y - z$ plane at $t = t_{opt}$ is presented in (c).

$$\hat{T} = \hat{T}_0 - \hat{v}_0 \frac{d\bar{T}}{dy} t. \quad (8.3)$$

Here, subscript ‘0’ denotes the initial condition. Eq. 8.1 implies that the wall normal disturbance velocity transports (lifts up) the mean momentum riding on its gradient to produce streamwise disturbance velocity. For a compressible flow, along with the streamwise momentum, perturbation temperature, and density are also conserved when the wall-normal velocity lifts the fluid particles along the shear layer (see Eqs. 8.2 and 8.3). Though the integrated value of streamwise momentum grows as a linear function of time, it is not necessary that the magnitude of disturbances will increase because they can also spread in time. Perturbations remain bounded as $t \rightarrow \infty$ and streaks in velocity and temperature form since the disturbed region in the stream-

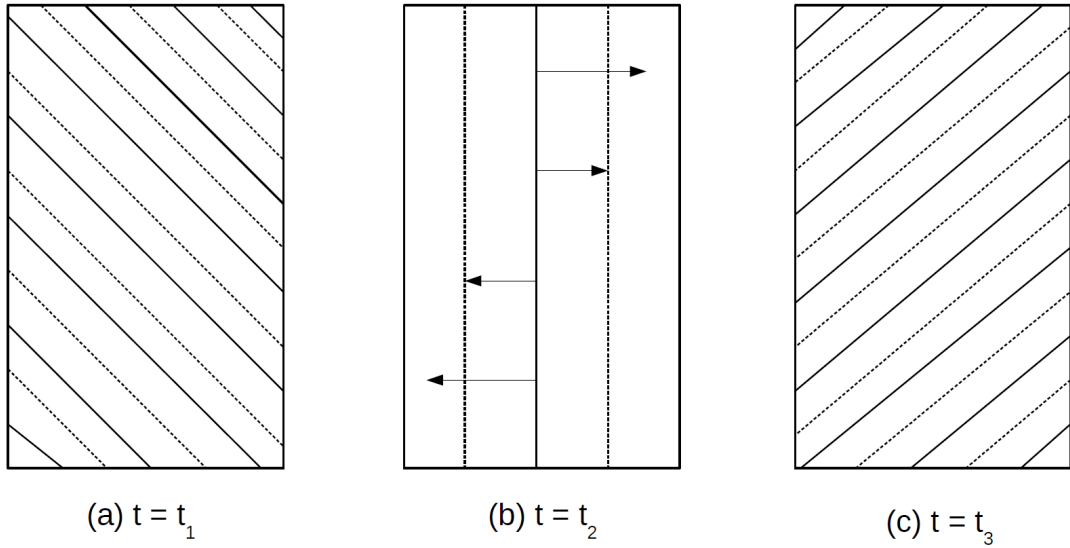


Figure 8.2: Rotation of a plane wave by the mean flow shear with time, the direction of mean flow is denoted by arrowheads. Solid and dashed lines indicate the high and low values of perturbation vorticity respectively.

wise direction grows with time. The disturbances which are elongated in the streamwise direction and comprise of positive and negative values of streamwise perturbation velocity are known as the streaks. Streaks give rise to localized regions, where flow alternately accelerates and decelerates, and also induce distinct regions of low and high temperature. It is reported that the amplitude of $\hat{u}$ actually has a logarithmic growth with time although transient energy grows linearly as $Re \rightarrow \infty$.

To demonstrate the lift-up mechanism, we plot the shape of initial condition along with the optimal disturbance for the κ -stratified flow at $M = 10$, $Re = 10^5$ and $Pr = 0.2$ are presented in Fig. 8.1. The spanwise wavenumber of 3.36 considered here corresponds to the maximum transient energy gain. The absolute values of the disturbances are plotted along the wall-normal direction in Figs. 8.1a and 8.1b. We notice that the optimal disturbance before amplification consists of mainly spanwise and normal components of velocity, other fluctuations are negligibly small. After amplification, the disturbance is mostly composed of streamwise velocity, temperature, and density fluctuations, other fluctuations are relatively small in magnitude (see Fig. 8.1b). The velocity vectors plotted in the $y - z$ plane corresponding to the amplified optimal disturbance shows the formation of two counter-rotating streamwise independent vortices, which lift the low-momentum and high-temperature flow up, from the adiabatic bottom wall, and push down the high-momentum, low-temperature flow, causing the formation of streaks in velocity and temperature.

8.2 Orr mechanism

For a spanwise independent perturbation ($\beta = 0$), the amplification of initial disturbance energy is caused by the Orr mechanism. The mechanism is explained in Fig. 8.2, where the rotation of a plane wave is shown with time. Initially, at time $t = t_1$, the vorticity pattern has a tilt opposite to the mean shear direction. By $t = t_2$, the lines of constant phase rotate until the angle is completely vertical (see Fig. 8.2b). The pattern continues to rotate until it aligns with the direction of mean shear (Fig. 8.2c). It is to be noted that when the tilt is vertical, the separation distance is maximum between the centers of low and high vorticity. To keep the vorticity conserved, the pattern has to rotate such that vorticities corresponding to the centers remain same. Therefore, in the amplifying state of the Orr mechanism, velocity perturbations must increase. Mathematically, in terms of perturbation energy density, we can write,

$$\frac{\partial E}{\partial t} = - \int_{y=0}^{y=1} \overline{uv} \frac{d\bar{U}}{dy} dy = - \int_{y=0}^{y=1} \overline{\frac{\partial y}{\partial x} \left(\frac{\partial \psi}{\partial y} \right)^2} \frac{d\bar{U}}{dy} dy, \quad (8.4)$$

here, the overbar denotes the averaged value. The above equation predicts that energy amplification ($\frac{\partial E}{\partial t} > 0$) can occur only when $\overline{\frac{\partial y}{\partial x} \frac{d\bar{U}}{dy}} < 0$, or the phase is oriented in the opposite direction to mean shear. Advection with the flow makes the disturbance phase tilted in the direction of shear, i.e., $\overline{\frac{\partial y}{\partial x} \frac{d\bar{U}}{dy}} > 0$, the energy of the fluctuation goes back to the mean flow.

8.3 Combined Orr and lift-up mechanism

As mentioned before, for a spanwise and streamwise dependent disturbance, both the lift-up and Orr-mechanism together decide the transient energy growth, which is the case for a μ - κ -stratified Couette flow. In this case, the disturbances primarily gain energy using the lift-up mechanism but obtain an additional increment of growth due to the energy transfer from the mean flow, which occurs via Reynolds stress, and coupling of the wall-normal velocity with temperature and density fluctuations due to compressibility.

In the case of μ - κ -stratified flow, we observe a spanwise alternating streamwise vortex, which is opposite to the singly-stratified cases, where we observe pure streamwise vortices as the optimal patterns (see Fig. 8.1c). To visualize the optimal disturbance in a μ - κ -stratified flow, we plot the streamwise and wall-normal components of velocity in the (x, y) plane at $M = 10$

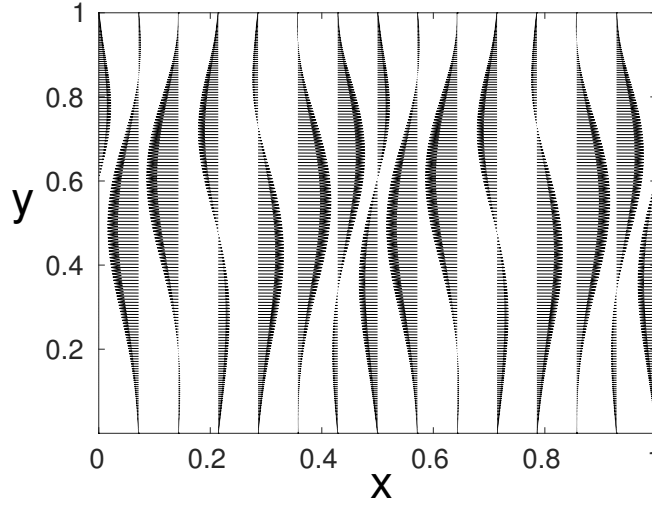


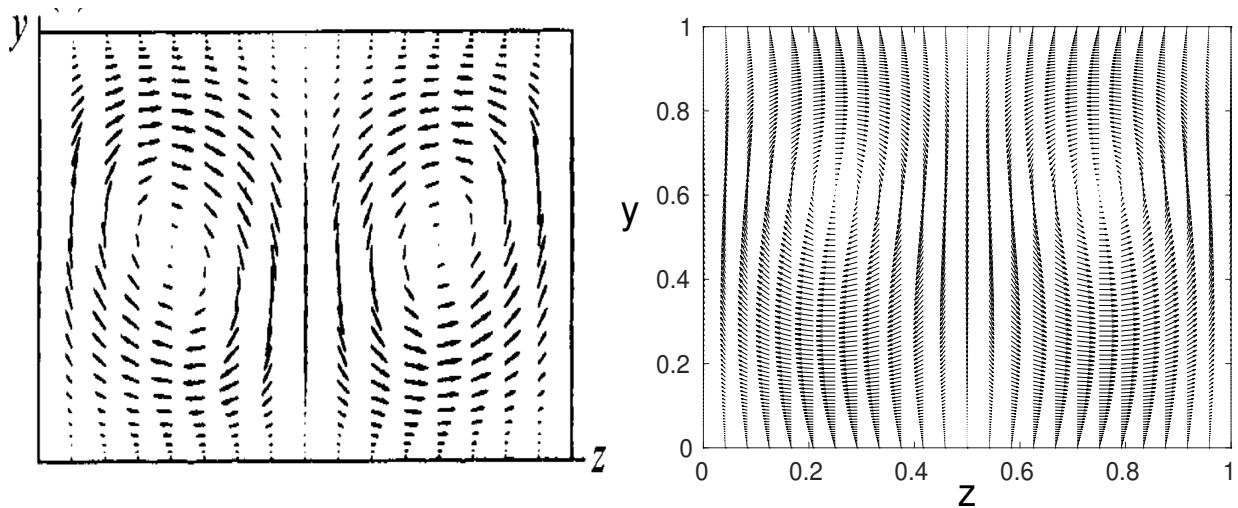
Figure 8.3: Optimal disturbance for a $\mu - \kappa$ -stratified Couette flow at $M = 10$, $Re = 10^5$, $\alpha = 0.05$, $\beta = 3.5$, and reference $Pr = 0.2$.

and $Pr = 0.2$ (see Fig. 8.3). In addition to its streamwise dependence nature, the vortices are also not parallel with the mean shear direction which was the case for the vortices observed in the μ -stratified and κ -stratified flows. Similar to the vorticity pattern shown in Fig. 8.2a (lines of constant phase tilted opposite to the shear), the vortices in a μ - κ -stratified flow also show a similar structure at the initial phase ($t = 0$). In this phase, the energy is extracted from the mean shear by transporting momentum down the mean momentum gradient (through Orr-mechanism) while trying to rise to an upright position as shown in Fig. 8.2b. The vortices continue to rotate until they become parallel to the direction of mean shear, similar to the pattern shown in Fig. 8.3 as well in Fig. 8.2c. The energy which was gained in the initial phase (amplification phase) (Fig. 8.2a) is now returned to the mean flow as the vortices align with the shear to attain a structure similar to Fig. 8.3.

8.4 Discussion

In this section, we will discuss and validate our compressible Couette flow results in the context of available experimental work and predictions of direct numerical simulation. A brief comparison with the relevant linear stability results is also presented.

To understand shear turbulence and sub-critical transition process, plane Couette flow (PCF) has been extensively used as a canonical geometry in the theoretical analysis of flow stability. Although for all Reynolds numbers incompressible Couette flow is linearly stable [74], it undergoes a sub-critical transition to the turbulent state. There appears a disorganized phase abruptly



(a) Figure taken from [87]

(b) Our result

Figure 8.4: The velocity vectors of wall-normal and spanwise velocities are plotted in the cross-flow plane ($y - z$) representing a streamwise vortex.

within the laminar state. Two experimental teams in Saclay and Stockholm ([75] and [76]) first observe the sub-critical behavior of the transition process in plane Couette flow and also characterized the turbulent spots [77]. Lundbladh and Johansson [78] are the first ones to carry out direct numerical simulation to examine the development of turbulent spots in PCF. They report that if the Reynolds number is around 375, the turbulent spots can sustain in the flow. One salient characteristic of Couette flow in the transition regime is that there exist alternate inclined stripes of laminar and turbulent flow. Prigent et al. [79], [80] observe these patterns in their experiments and later Barkley and Tuckerman [81], Duguet et al., [82] and Tuckerman and Barkley [83] and many others replicated numerically. Couliou and Monchaux [85] recently perform experiments to study the growth dynamics of turbulent spots in PCF. Using particle image velocimetry, they also investigate the large-scale flows in Couette flow, which are seen when turbulent and laminar regions exist together [84]. Couliou and Monchaux also notice that these large-scale structures grow prior to the development of the turbulent spot and play a vital role in the emergence of organized patterns. Streaks which surround the growing turbulent spots are recognized as an important element in the laminar to turbulent transition process. These longitudinal structures emanating from pairs of counter-rotating streamwise vortices have also been studied experimentally by Dauchot and Daviaud [77] and Bottin et al. [86].

The dynamics of flow pattern near the wall region of a turbulent incompressible Couette flow has been studied by Hamilton et al. [87] using direct numerical simulation. They notice that the

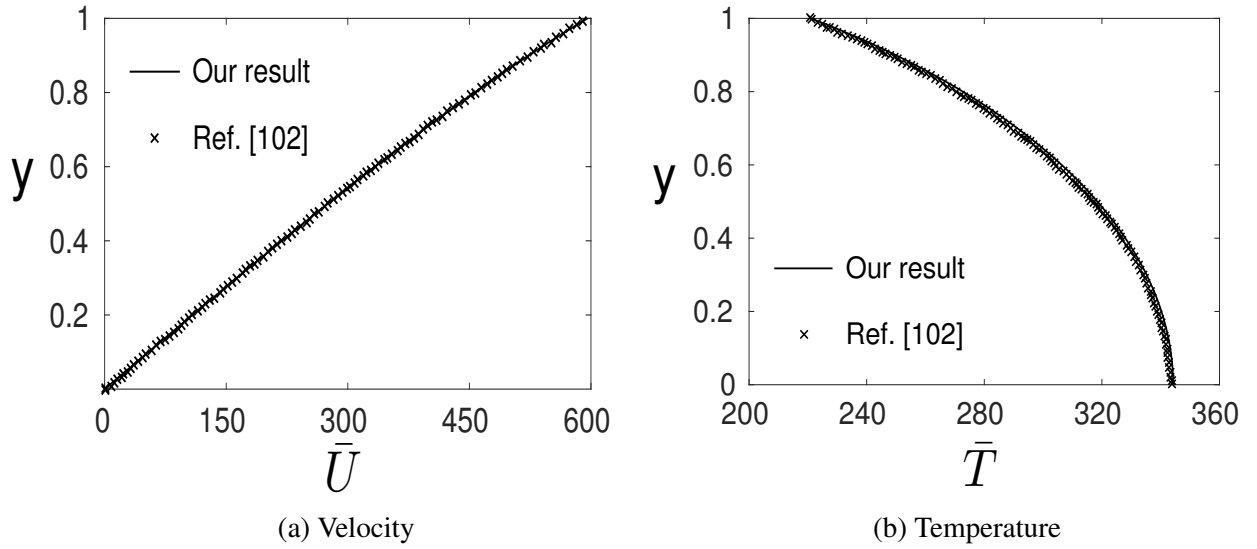
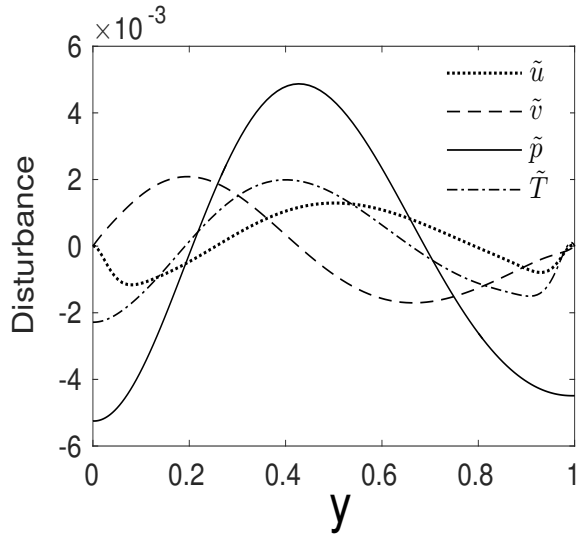


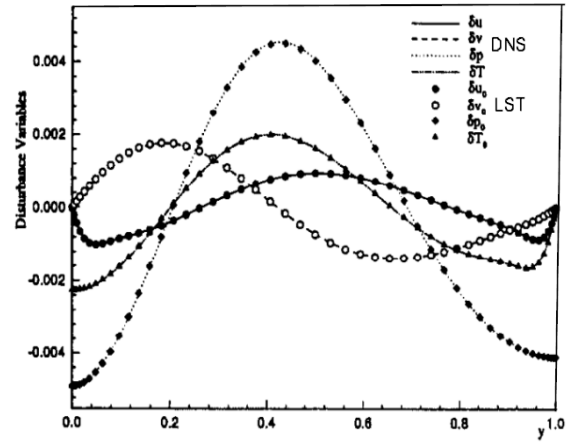
Figure 8.5: The variation of mean flow velocity and temperature profiles from our calculation is compared with the direct numerical simulation of Dong and Zhong [102] at $M = 2$ and $Pr = 0.72$.

regeneration process of these structures is well-defined and also spatially organized and quasi-cyclic in nature. At the beginning of this self-sustaining process, streamwise vortices create the elongated streaks. Subsequently, the streamwise rolls give rise to a spanwise modulation of the tangential velocity, which creates an inflectional instability. This causes the flow to break down and in the final step of the regeneration cycle, streak instabilities become non-linear and they feedback on the downstream streamwise rolls. Fig. 8.4 shows a comparison of the streamwise vortices observed in our flow (κ -stratified, μ -stratified and constant shear flow) with that observed in the numerical computation of Hamilton et al. [87]. To visualize the vortices, we plot the velocity vectors corresponding to spanwise and wall-normal velocities in Fig. 8.4b and notice that the structure of the streamwise vortices matches well with the pattern shown in Fig. 8.4a, which is taken from the direct numerical simulation of Hamilton et al. [87]. The mechanism by which the streaks are created from the vortices is known as the lift-up effect, and it leads to short-term amplification of disturbance energy. Therefore, even in a compressible Couette flow, the physical mechanism leading to non-modal energy growth is the same as that observed in an incompressible Couette flow or some other shear flows ([87]-[92]).

Due to the difficulties experienced while conducting and taking the measurements, the experimental situation with regard to the compressible plane Couette flow is far less satisfactory. In our knowledge, there is no experimental work till late which studies the transition phenomenon in a high Mach number Couette flow. Therefore, we have to resort to direct nu-



(a) Our result



(b) Taken from Ref. [102]

Figure 8.6: Comparison of instantaneous flow perturbations computed using linear stability analysis at $M = 2$, $Re = 1000$, $Pr = 0.72$ and $\alpha = 3$, with the DNS and LST results of Dong and Zhong [102]. A good qualitative match can be seen between our result and the plot taken from in Ref. [102].

merical simulation (DNS) to validate as well as examine the transition process in a compressible Couette flow. To validate the base flow results, we plot the mean velocity and temperature profiles along the wall-normal direction for a $M = 2$ flow in Fig. 8.5. The top wall is maintained at a temperature of 220.67 K, whereas the adiabatic boundary condition is imposed at the bottom wall. These parameters and boundary conditions are same as that considered in Dong and Zhong [102], such that we can have a direct comparison of the results. From Fig. 8.5, we notice that our mean flow results have a good agreement with the numerical computation of Ref. [102]. Next, we compare the instantaneous flow perturbations obtained from our linear stability analysis with that of temporal DNS results, and also LST calculations of Dong and Zhong [102]. The base flow is the same as that presented in Fig. 8.5 and for stability calculations, we have considered a wavenumber and Reynolds number of 3 and 1000 respectively. The instantaneous flow perturbations for streamwise velocity, normal velocity, pressure, and temperature are plotted along the wall normal direction in Fig. 8.6. The qualitative behavior of the flow variables obtained from our calculation (Fig. 8.6a) agrees well with the DNS results shown in Fig. 8.6b.

In addition to DNS studies, linear stability analysis is extensively used to investigate the stability of supersonic Couette flows ([20] - [22], [67], [93] - [96]). Most of these studies are dedicated to examine the effect of viscosity variation on different instability modes for a Couette flow with perfect gas or a vibrationally excited gas. Since a significant part of our research work

deals with viscosity stratification, we compare our results with Malik et al. [22], [67] and Hu and Zhong [20]. The eigenspectrum, variation of the growth rate of mode 1 and 2 with wavenumber and critical Reynolds number, at varying Mach numbers are compared and found an excellent match (see Appendix 3).

Investigating both modal and non-modal instability, Malik et al. [22] report that stratification of viscosity delays the transition process in a compressible Couette flow. We also observe a strong stabilizing effect of viscosity stratification, which causes a significant increase in the critical Reynolds number (see section 6.2.1). Similar to the prediction of Malik et al. [67], we notice a decrease in the non-modal energy growth with an increase in the Mach number. This is caused by the reduction in the transfer of mean flow energy to the disturbances. Various constituent terms of the non-modal energy budget are also compared, along with the transient energy amplification for a compressible Couette flow with viscosity and conductivity stratification, and found a striking match. We also found that the physical mechanism which causes short-term energy growth in a viscosity and conductivity stratified Couette flow is (a combination of lift up and Orr mechanism) same as that reported in Malik et al. [22], [67] (For more details visit section 8.3).

8.5 Summary

In this chapter, we discuss various physical mechanisms leading to the amplification of disturbance energy for a finite period of time. For most of the model problems considered, we notice that transient energy growth is caused by the lift-up mechanism. For example, in the case of reference, averaged, κ -stratified and μ -stratified flows, where optimal disturbances are streamwise independent, lift-up mechanism leads to the maximum temporal energy amplification. However, in the case of μ - κ -stratified flow, 3-D optimal disturbances with non-zero α and β lead to the highest transient energy amplification. This can be attributed to the interplay of both the vortex-tilting and Reynolds stress mechanism. In this case, pure streamwise vortices are not the optimal patterns, and they have a phase tilt opposite to mean shear direction. At the end of the chapter, we also discuss and validate our compressible Couette flow stability results with the available experimental and DNS works.

Chapter 9

Flat plate boundary layer

In the first part of the report, we have investigated the effect of variation in the transport properties on the modal stability and transient growth of a compressible Couette flow. It is found that the changes in the transport properties can be quantified in terms of the Prandtl number. In this chapter, we analyze the role of the Prandtl number on the linear stability of a high-speed boundary layer flow over a flat plate. For this, we compute the self-similar compressible boundary layer profiles at various Mach, Prandtl, and Reynolds numbers and a temporal linear stability analysis is performed.

9.1 Base flow

A steady two-dimensional compressible laminar boundary layer flow over a flat plate is solved using the Levy-Lees similarity transformation. In this approach, the coordinates are transformed from the physical domain (x, y) to a transformed space (ξ, η) which are related by,

$$\xi = \int_0^x \rho_e \mu_e U_e dx, \quad (9.1)$$

$$\eta = \frac{U_e}{\sqrt{2\xi}} \int_0^y \rho dy, \quad (9.2)$$

where ρ_e , U_e and μ_e are the density, velocity, and viscosity at the boundary layer edge. Using the boundary layer approximations along with the above transformation, the Navier-Stokes equations reduce to a system of two coupled ODEs given by,

$$(C f'')' + f f'' = 0, \quad (9.3)$$

$$\left(\frac{C}{Pr}g'\right)' + fg' + (\gamma - 1)M^2Cf''' = 0, \quad (9.4)$$

where $f' = U^*/U_e$ is the non-dimensional velocity, $g = T^*/T_e$ is the non-dimensional temperature, C is the Chapman-Rubesin factor which is the viscosity-temperature ratio in their non-dimensional forms, M is the edge Mach number, Pr is the local Prandtl number and $\gamma = 1.4$ is assumed. The asterisk and the subscript “e” denote dimensional and reference quantities (boundary layer edge values), respectively, and prime (also D) refers to the derivative with respect to the transformed wall normal direction, η . Viscosity is calculated using Sutherland’s law, which can be expressed in the non-dimensional form as follows,

$$\mu = T^{\frac{3}{2}} \frac{(1+B)}{(T+B)}, \quad (9.5)$$

where $B = 110.4/T_e$ and T_e is in Kelvin. Further, conductivity is calculated by assuming a constant value of Pr , based on the free-stream parameters. This constant value of Pr is systematically varied to investigate the effect of variation in Prandtl number on flow stability.

Eqs. 9.3 and 9.4 are subjected to the following boundary conditions at the wall ($\eta = 0$),

$$f = 0, f' = 0 \text{ and } g = T_w \text{ or } g' = 0, \quad (9.6)$$

where the temperature boundary condition depends on whether the wall is isothermal or adiabatic. At the free-stream ($\eta \rightarrow \infty$), the boundary conditions are,

$$f' \rightarrow 1 \text{ and } g \rightarrow 1. \quad (9.7)$$

In the above equation, ∞ represents the free-stream values. The base flow (Eqs. 9.3 and 9.4) together with the boundary conditions (Eqs. 9.6 and 9.7) can be solved as an initial value problem with an initial guess for f'' and g at the wall (for an adiabatic wall). The actual values of f'' and g are obtained using the iterative shooting method which ensures that these values satisfy the boundary conditions at the free-stream. Note that in order to convert the variable η back to the physical domain y , we use the inverse transformation given by,

$$\int_0^\eta \sqrt{2T(\eta)} d\eta = \int_0^y \frac{1}{x} \sqrt{Re_x} dy. \quad (9.8)$$

The length scale used for the definition of the Reynolds number is $\sqrt{\frac{x\nu_e}{u_e}}$.

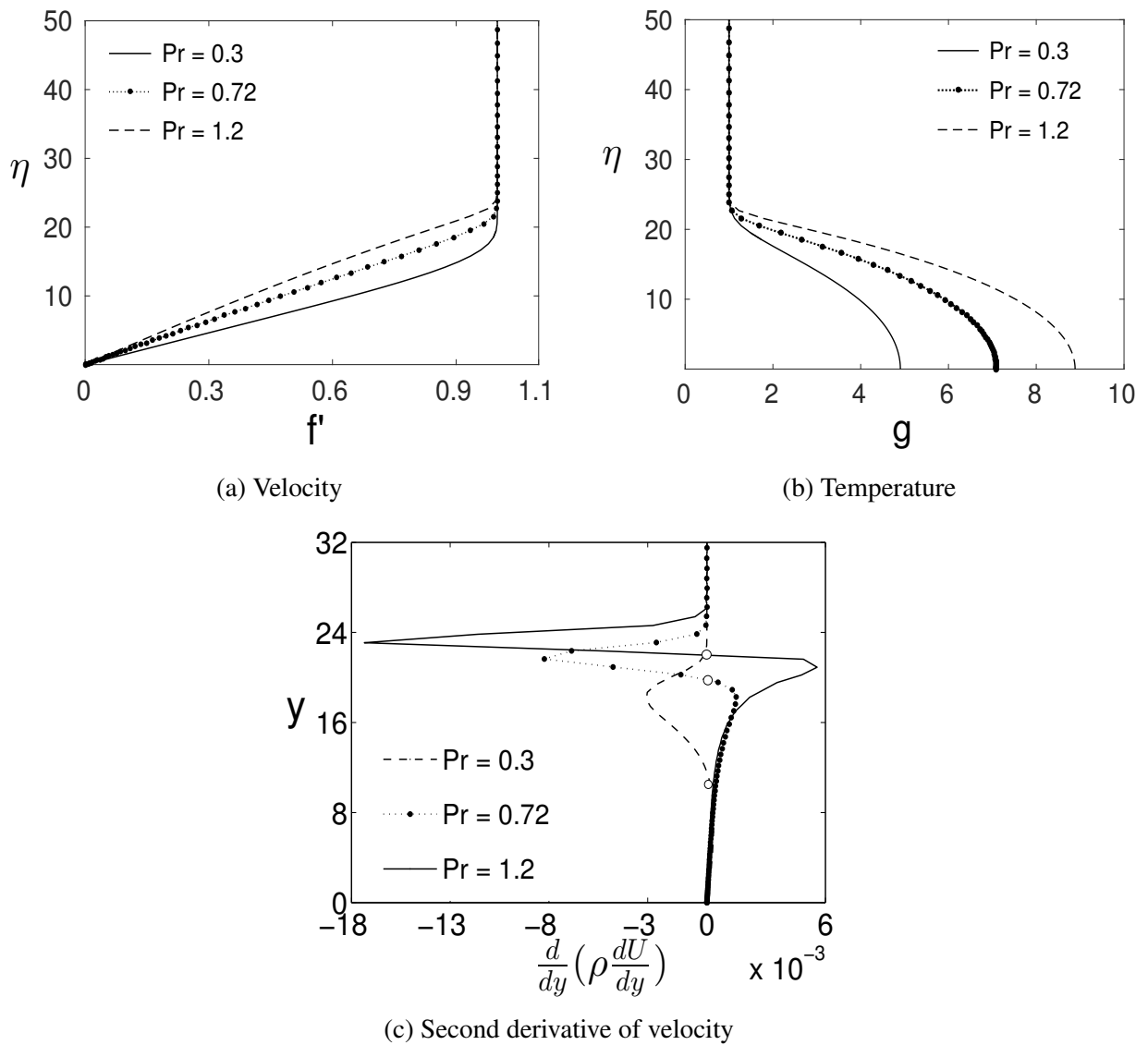


Figure 9.1: Effect of the Prandtl number on the mean flow at $M = 6$.

The mean velocity and temperature profiles for three Prandtl numbers (0.3, 0.72 and 1.2) at $M = 6$ and stagnation temperature of $T_o = 300\text{K}$ are shown in Figs 9.1a and 9.1b respectively. As the Prandtl number is increased, the thickness of the boundary layer increases due to the increase in viscosity (Fig. 9.1a). With an increase in Pr , the wall temperature also increases significantly, since the recovery factor scales as $\sqrt{Pr}$. The second derivative of mean flow velocity is plotted in Fig. 9.1c, and the location where $D(\rho DU)$ crosses zero, is called the generalized inflection point (GIP). The presence of the generalized inflection point is an indication of first mode instability. With an increase in the Prandtl number, the generalized inflection point shifts close to the boundary layer edge (see the open circles in Fig. 9.1c). This implies that the inflectional instabilities move outwards for the higher Prandtl number cases.

9.2 Numerical method

The linear stability equations are solved using the Chebyshev spectral collocation method. The collocation points y_c are given by,

$$y_{c_j} = \cos \frac{\pi j}{N}, \quad j = 0, 1, \dots, N. \quad (9.9)$$

The domain $\eta = 0$ to $\eta = \eta_{max}$ is mapped to the Chebyshev domain of $y_c \in [-1, 1]$ through the following transformation that accounts for grid stretching:

$$\eta = a \frac{1 + y_c}{b - y_c}, \quad (9.10)$$

$$a = \frac{\eta_i \eta_{max}}{\eta_{max} - 2\eta_i}, \quad \text{and} \quad b = 1 + \frac{2a}{\eta_{max}}. \quad (9.11)$$

In the above equation, η_i is a parameter which is used to cluster half of the Chebyshev points near the wall. The choice of this parameter is made such that there is a sufficient clustering within the boundary layer around the inflection point.

We have carried out a temporal linear stability analysis similar to the previous Couette flow studies. This implies that the disturbance frequency is complex and the wavenumber is real. The imaginary part of the frequency provides the amplification rate of the perturbations. Boundary conditions applied to the perturbations are,

$$\hat{u}(0) = \hat{u}(\infty) = 0, \quad \hat{v}(0) = \hat{v}(\infty) = 0, \quad \hat{w}(0) = \hat{w}(\infty) = 0; \quad (9.12)$$

$$\hat{T}(\infty) = 0, \quad D\hat{T}(0) = 0 \quad \text{or} \quad \hat{T}(0) = 0. \quad (9.13)$$

We replace the rows of A and B (from the eigenvalue problem $B\psi = \omega A\psi$) to impose the proper boundary conditions on the eigenfunctions. For more details on the boundary conditions please refer to section 3.2.1.

9.3 Linear stability results

9.3.1 Identification of modes in the eigenspectrum

Fig. 9.2 presents the temporal eigenvalue spectra computed by solving the generalized eigenvalue problem using the global method. The real (c_r) and imaginary (c_i) parts of the phase velocity are plotted for a flow over a flat plate at $M = 6$, $Re = 6000$ and $Pr = 0.4$. A

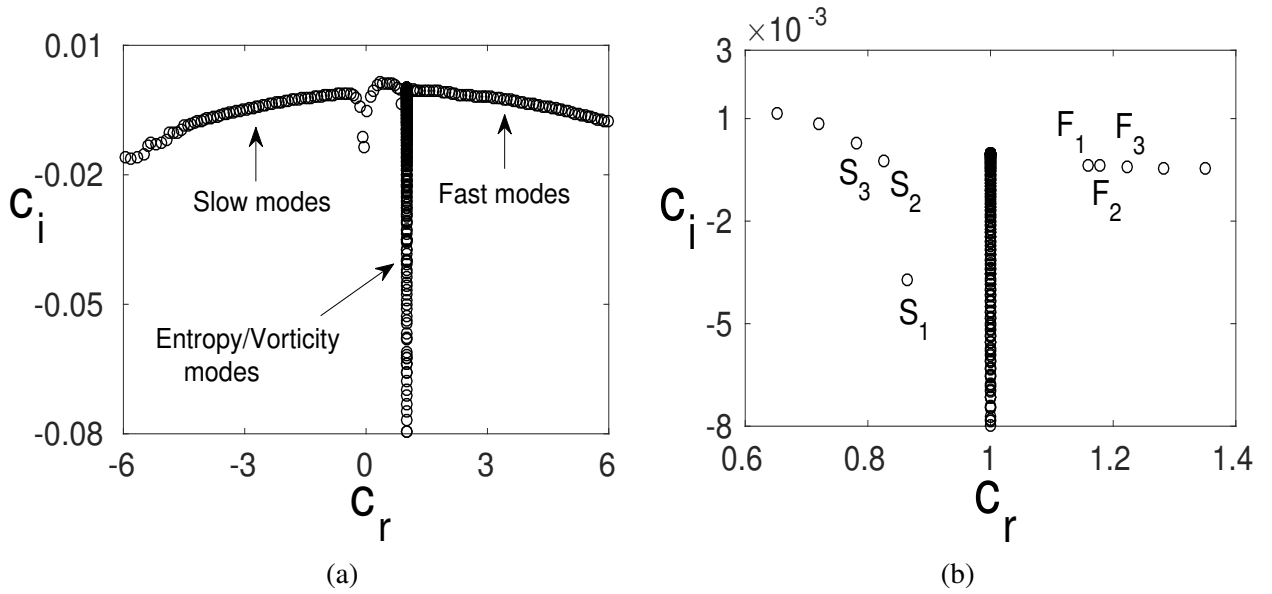


Figure 9.2: Global eigenvalue spectra computed for a boundary layer flow over a flat plate at $M = 6$, $Re = 5000$, $Pr = 0.4$ and $\alpha = 0.03$. The reference temperature considered is 300K. (a) Three different continuous branches are shown, which include slow and fast acoustic modes, and the vorticity and the entropy branch near $c_r = 1$. (b) The magnified view of (a) close to the vorticity/entropy branch showing a few discrete modes. Slow and fast modes are marked as S and F respectively.

two-dimensional disturbance is considered with a stream-wise wavenumber of 0.03.

In compressible boundary layers, the continuous spectrum consists of seven branches [69], out of which two branches of fast and slow acoustic modes propagate downstream. Two other branches which also travel downstream composed of entropy and vorticity modes. The remaining branches correspond to highly-damped, upstream propagating waves. The eigenspectrum in Fig. 9.2a shows a discrete approximation of the continuous spectra, where we notice three different branches. By looking at their characteristics outside the boundary layer, the discrete modes can be differentiated from the continuous spectrum. While the modes in the continuous spectrum need to be bounded, the discrete modes should decay at the free-stream. Two branches extending to the left and right in Fig. 9.2a propagate as acoustic waves and are known as fast and slow modes depending on their phase speed value. The vertical branch corresponds to the entropy and vorticity waves and they travel with the mean flow velocity ($c_r = 1$).

To identify a few discrete modes, we have presented a magnified view of the eigenspectrum near $c_r = 1$ in Fig. 9.2b. The first three fast and slow modes are marked as F_1, F_2, F_3 and S_1, S_2, S_3 respectively. The first slow and fast modes are identified from their behavior at the leading edge of the flat plate as $\alpha \rightarrow 0$. The slow discrete mode synchronizes with the slow mode of the continuous spectrum with a phase speed of $1 - \frac{1}{M}$. Similarly, the phase speed of

the fast discrete mode tends to $1 + \frac{1}{M}$ of the fast acoustic wave in the long wavelength limit. As the wavenumber is increased, the fast discrete mode F^- synchronizes with the continuous branch near $c_r = 1$ and departs from the branch as another discrete mode F^+ . With a further increase in the disturbance wavenumber, the phase speeds of mode F^+ and the slow mode come closer and there is a possibility of synchronization, which leads to increased growth rates. The Mack mode instabilities that appear in supersonic boundary layers are due to the synchronization between fast and slow modes, and the second mode instability is due to the synchronous interaction between the first slow, and the fast mode F^+ . The branching pattern associated with the synchronization between the modes is a function of the free-stream parameters such as the Prandtl number, Mach number, and the wall-temperature ratio. In the next section, we discuss the role of Prandtl number on the branching pattern as the dispersion curves branch out.

9.3.2 Validation and sensitivity analysis

Fig. 9.3 shows the sensitivity of various modes in the eigenspectrum to the selection of the boundary layer edge location (η_{max}) and the number of Chebyshev points (N). We have considered a boundary layer flow at $M = 3$, $Pr = 0.7$, $Re = 3000$ and $T_o = 300\text{K}$. A oblique disturbance with $\alpha = 0.06$ and $\beta = 0.1$ is provided. We notice that an increase in the number of Chebyshev points affects the fast-decaying vorticity/entropy modes significantly (see Fig. 9.3a). However, the discrete modes converge for a moderate number of Chebyshev modes. A very low number of Chebyshev points can lead to the vorticity and entropy modes getting closer to the acoustic branch, which in turn leads to the spurious interactions between these modes. Therefore, care must be taken to ensure that a sufficient number of points are used to avoid such spurious modes. Further, Fig. 9.3b shows that the choice of η_{max} affects the higher modes of the two acoustic branches of the eigenspectrum. The first fast and slow modes are not affected

M	Re	T_o (K)	α	β	ω (Ref. [73])	ω (our code)
0.5	2000	277.78	0.1	0	$0.02908180 + 0.00224419i$	$0.0290679933 + 0.0022429088i$
2.5	3000	333.33	0.06	0.1	$0.0367339 + 0.0005840i$	$0.0367477188 + 0.0005883717i$

Table 9.1: Comparison of the eigenvalue of the most unstable mode for two test cases from Ref. [73] at $Pr = 0.7$.

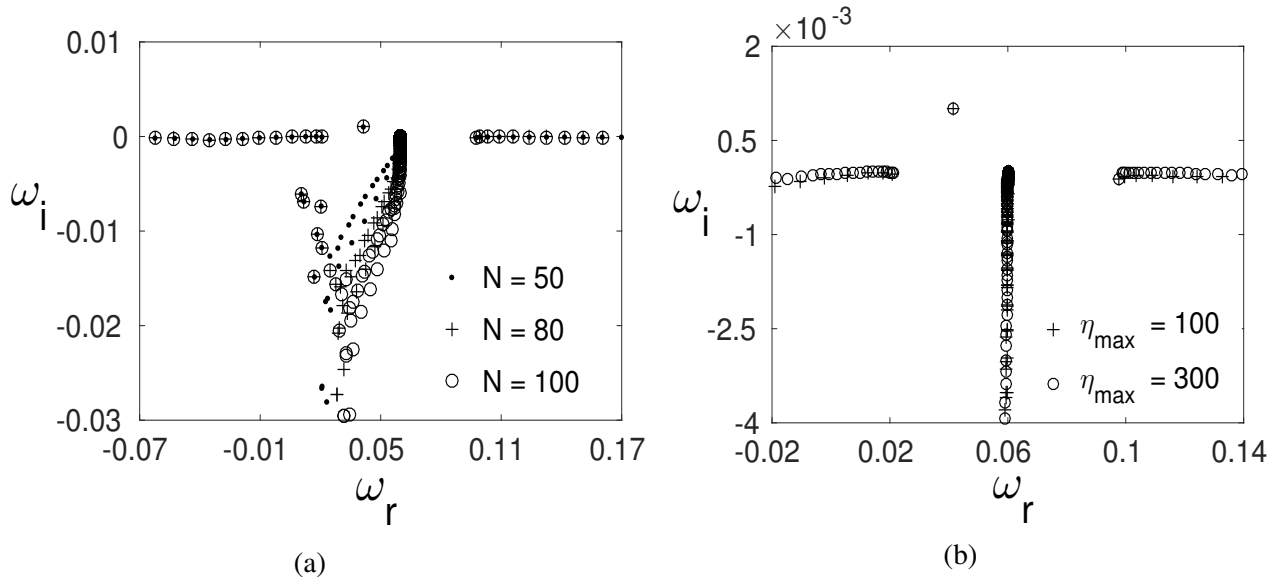


Figure 9.3: Sensitivity of the eigenspectrum to the (a) number of points (N); (b) the location of boundary layer edge (η_{max}); for $M = 3$, $Pr = 0.7$, $Re = 3000$, $\alpha = 0.06$, $\beta = 0.1$ and $T_o = 300K$ boundary layer flow.

by the choice of η_{max} beyond a value of 100. However, the location of the higher modes in the acoustic branches can shift depending on the η_{max} value. A good choice of η_{max} will ensure that the fast and slow modes are accurately captured and that the higher modes do not lead to spurious instabilities. The height of the domain should be sufficiently larger than the boundary layer thickness, and care should be taken to place about half the grid points within the boundary layer. Also, in the case of supersonic disturbances, the eigenfunctions show an oscillatory behavior outside the boundary layer edge and decay very slowly. Therefore we have used a larger computational domain, which although increases the computational cost, ensures that the disturbances vanish in the far-field.

To validate our code, we have compared our results with the test cases mentioned in Ref. [73], and found a match up to fourth and the fifth decimal places while comparing the real and imaginary parts of the eigenvalues respectively (see Table 9.1). A two dimensional disturbance is considered ($\alpha = 0.1$) for the flow with a Mach number of 0.5, and a three dimensional perturbation ($\alpha = 0.06$, $\beta = 0.1$) is assumed for the $M = 2.5$ case. An insulated condition is imposed at the wall and a reference Prandtl number of 0.7 is considered.

9.3.3 Effect of Prandtl number on the slow and fast modes

For investigating the effect of Prandtl number on the fast and slow modes, the growth rates of both the modes are plotted in Fig. 9.4, for a flow over a flat plate at $M = 4$, $Re = 8000$, $\beta = 0$,

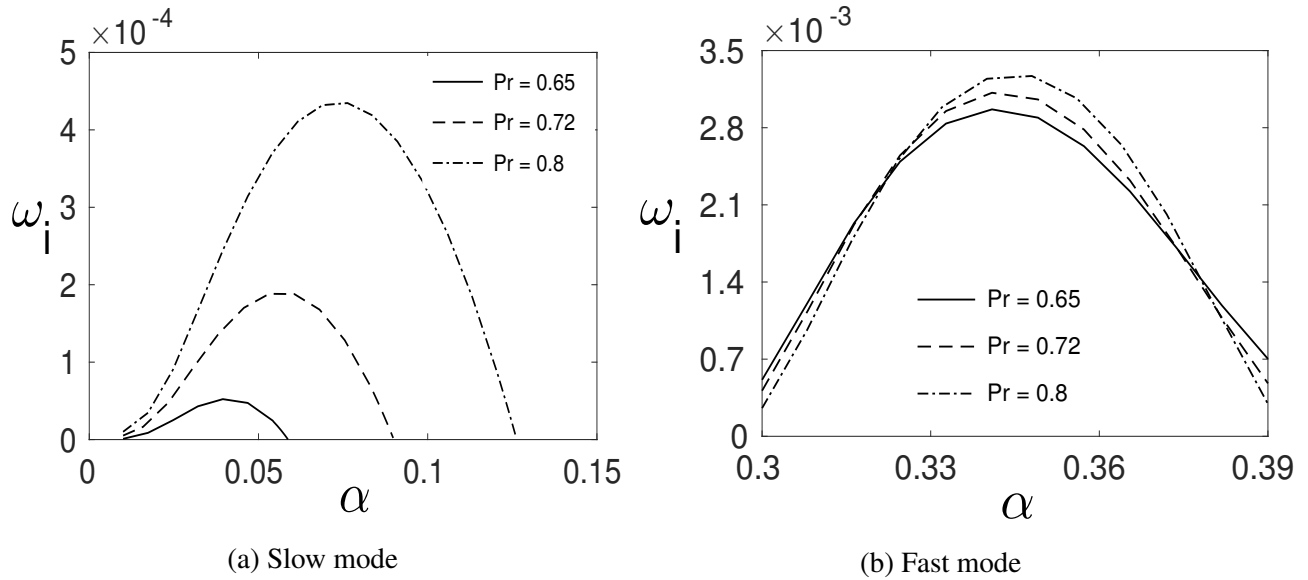


Figure 9.4: Effect of the Prandtl number on the growth rate of the fast and slow modes at $M = 4$, $Re = 8000$, $\beta = 0$ and a reference temperature of 288K.

and reference temperature of 288K, with three Prandtl numbers. Fig. 9.4a shows that increasing the Prandtl number has a significant destabilizing effect on the slow mode. An increase in Pr from 0.65 to 0.8 leads to an increase in the maximum growth rate of the slow mode by a factor of around 10. Further, the destabilizing effect of an increase in Pr is monotonic at each stream-wise wavenumber. The wave number corresponding to the peak in the growth rate of the slow mode shifts to higher values at higher Prandtl numbers.

The overall effect of increasing the Prandtl number is also to destabilize the fast mode, however, it is not monotonic. For instance, around a wavenumber of 0.3 and 0.32, it can be seen that the growth rate of the fast mode corresponding to $Pr = 0.65$ is the highest, followed by $Pr = 0.72$ and 0.8 respectively. However, the maximum growth rate is caused by the highest Prandtl number ($= 0.8$) and the trend is monotonic with Pr (see Fig. 9.4b).

9.3.4 Branching type as a function of the Prandtl number

Fig. 9.5 plots the variation of phase speed and growth rate with wavenumber for the first two discrete modes for a $M = 6$ and $Re = 5000$ compressible flow over a flat plate, at a stagnation temperature of 300K. The wavenumber is varied from 0.06 to 0.14 and the Prandtl number for the case considered is 0.39. When the wavenumber is increased from 0.06, the phase speed of the fast mode decreases monotonically and it synchronizes with entropy and vorticity modes of the continuous spectrum at a phase speed of around 1 (Fig. 9.5c). A small change in the growth

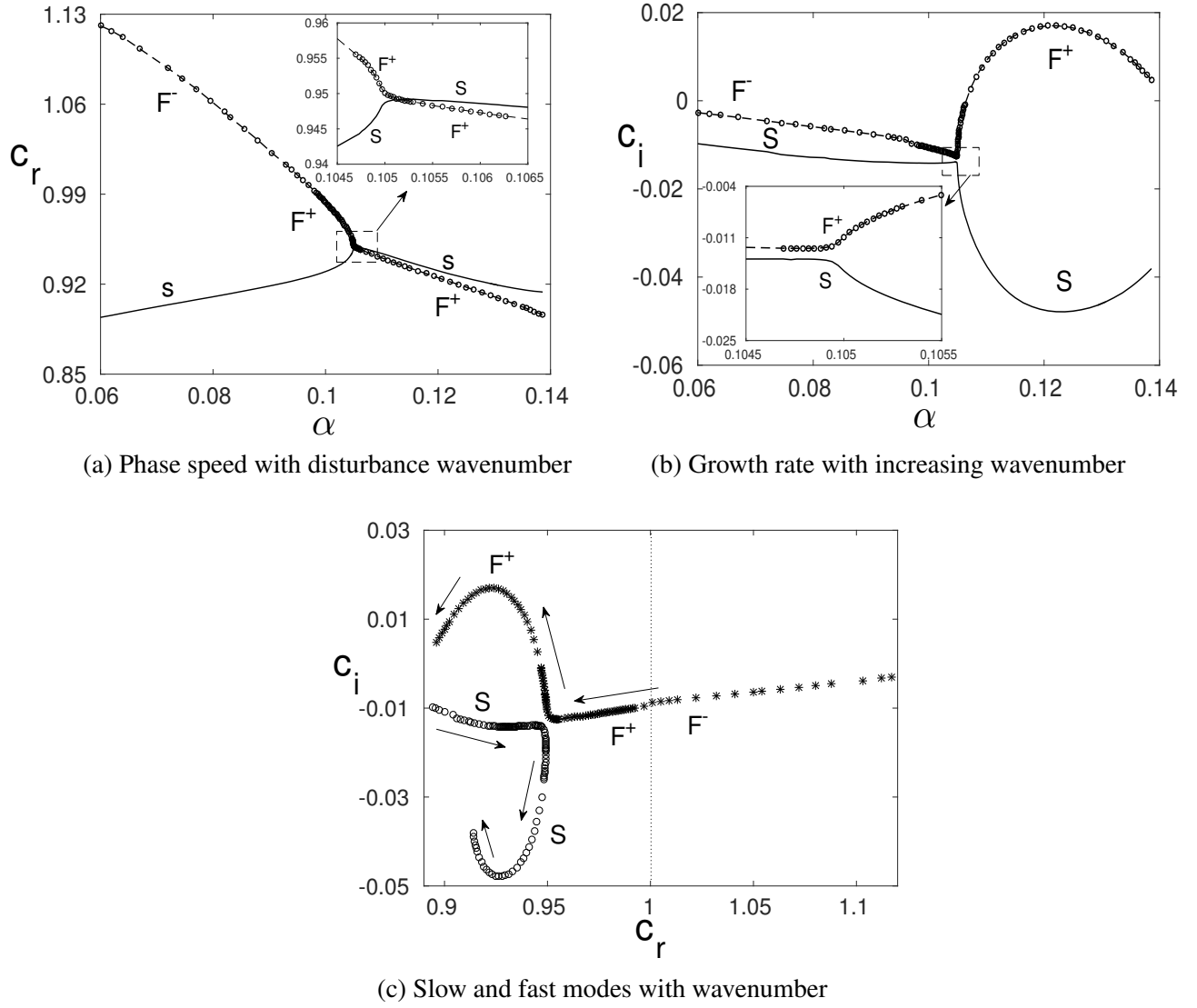


Figure 9.5: The phase speed and growth rate of the slow and the first fast acoustic modes are shown as a function of the disturbance wavenumber. The flow parameters considered are: $M = 6$, $Re = 5000$, $Pr = 0.39$ and $T_o = 300K$. The magnified view near the synchronization location (marked by a small rectangle with dashed lines) are shown in the insets. The direction of arrows indicates increasing wavenumber.

rate of the fast mode is observed after it crosses the entropy and the vorticity branch (near the dashed line in Fig. 9.5c). Therefore the fast mode before and after synchronization is considered as two distinct modes. The mode before the synchronization ($C_r > 1$) with the continuous branch is denoted as F^- and another one which appears directly after the synchronization ($C_r \rightarrow 1 - 0$) is known as F^+ . These two modes can be distinguished from their characteristics outside the boundary layer. While studying the initial value problem which represents the receptivity to entropy and vorticity waves, it is necessary to differentiate between the modes F^- and F^+ [69].

As the wavenumber is increased further, the fast mode F^+ whose phase speed decreases continuously, attain a similar phase speed as that of the slow mode (see the inset in Fig. 9.5a).

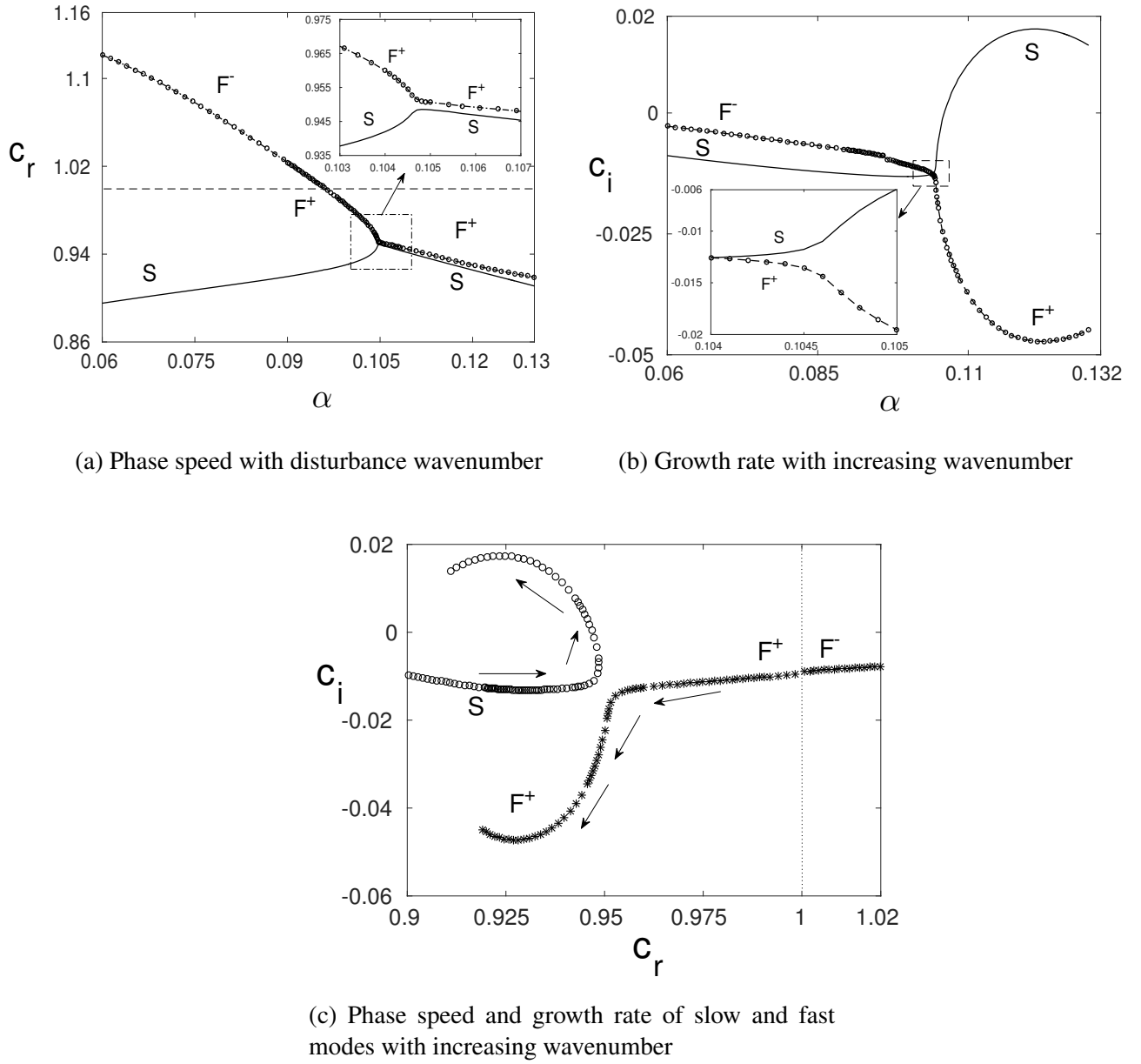


Figure 9.6: The variation of phase speed and growth rate of the first two discrete modes are shown with increasing wavenumber. The flow parameters considered here are same as those in Fig. 9.5, however, the Prandtl number is increased to 0.4. Increase in wavenumber is shown by the direction of the arrows.

At this location, we observe that the fast mode F^+ which has a higher growth rate than the slow mode before the synchronization (marked by a dashed rectangle in Fig. 9.5b), becomes more unstable due to the interaction with the slow mode. The unstable mode F^+ after the synchronization is known as the Mack's second mode. There is also a substantial stabilization of the slow mode (see Figs. 9.5b and 9.5c). With an additional increment in the disturbance wavenumber, the phase speed of the slow mode decreases (see the inset in Fig. 9.5a) and it crosses the fast mode. The fast mode F^+ continues to move to the left in the eigenspectrum (Fig. 9.5c), which implies that its phase speed decreases further. This branching pattern where

the fast mode becomes the dominant instability after the synchronization is observed only at relatively low Prandtl numbers (≤ 0.39 for this case).

As the Prandtl number is increased to 0.4, the branching pattern switches, with a change in the dominant instability mode from the fast to the slow acoustic mode. To show this, we have plotted the eigenvalues for the first two acoustic modes as a function of increasing wavenumber in Fig. 9.6c. The parameters in Fig. 9.6 are same as in Fig. 9.5, only the difference is in the Prandtl number value, which is 0.4 in the current case. As opposed to the previous case ($Pr = 0.39$), the first two discrete modes do not attain a common phase speed (see the inset in Fig. 9.6a). When the wavenumber is around 0.105, we observe that the fast mode F^+ attains a phase velocity close to the slow acoustic mode, however, the slow mode does not cross the fast mode. Similar to the previous case, the phase speed of the fast mode monotonically decreases with a further increase in the disturbance wavenumber. The variation of the growth rate of both the modes with wavenumber shows that, when the phase speed of the modes come closer, the slow mode which was more stable before the interaction, becomes the dominant instability after the synchronization and it manifests as the Mack's second mode. This type of branching pattern where the slow mode becomes the most unstable mode after the synchronization is observed for a high Prandtl number (≥ 0.4 at $M = 6$) boundary layer flow.

9.3.5 Eigenfunctions corresponding to the fast and slow modes

Fig. 9.7 plots the absolute values of the eigenfunctions for the first two discrete modes F^+ and F^- along the shear layer. The parameters considered here are: $M = 6$, $Re = 3000$, $T_o = 300K$ and $Pr = 0.4$. A two-dimensional disturbance with a stream-wise wavenumber of 0.05 is provided. The eigenfunctions are normalized by the absolute value of the pressure amplitude at the wall. The parameters, in this case, are such that the first two discrete modes are far ahead of the synchronization point and the fast mode is more unstable than the slow mode. The velocity eigenfunctions plotted in Fig. 9.7a shows the presence of viscous sublayer close to the wall ($0 < y < 1.4$). At a wall normal distance of around 20, the temperature eigenfunction has a peak (see Fig. 9.7b), which is also the location of the critical layer for both the modes. The pressure eigenfunctions are plotted in Fig. 9.7c and we notice that the slow mode has a higher magnitude for pressure perturbation compared to the fast mode.

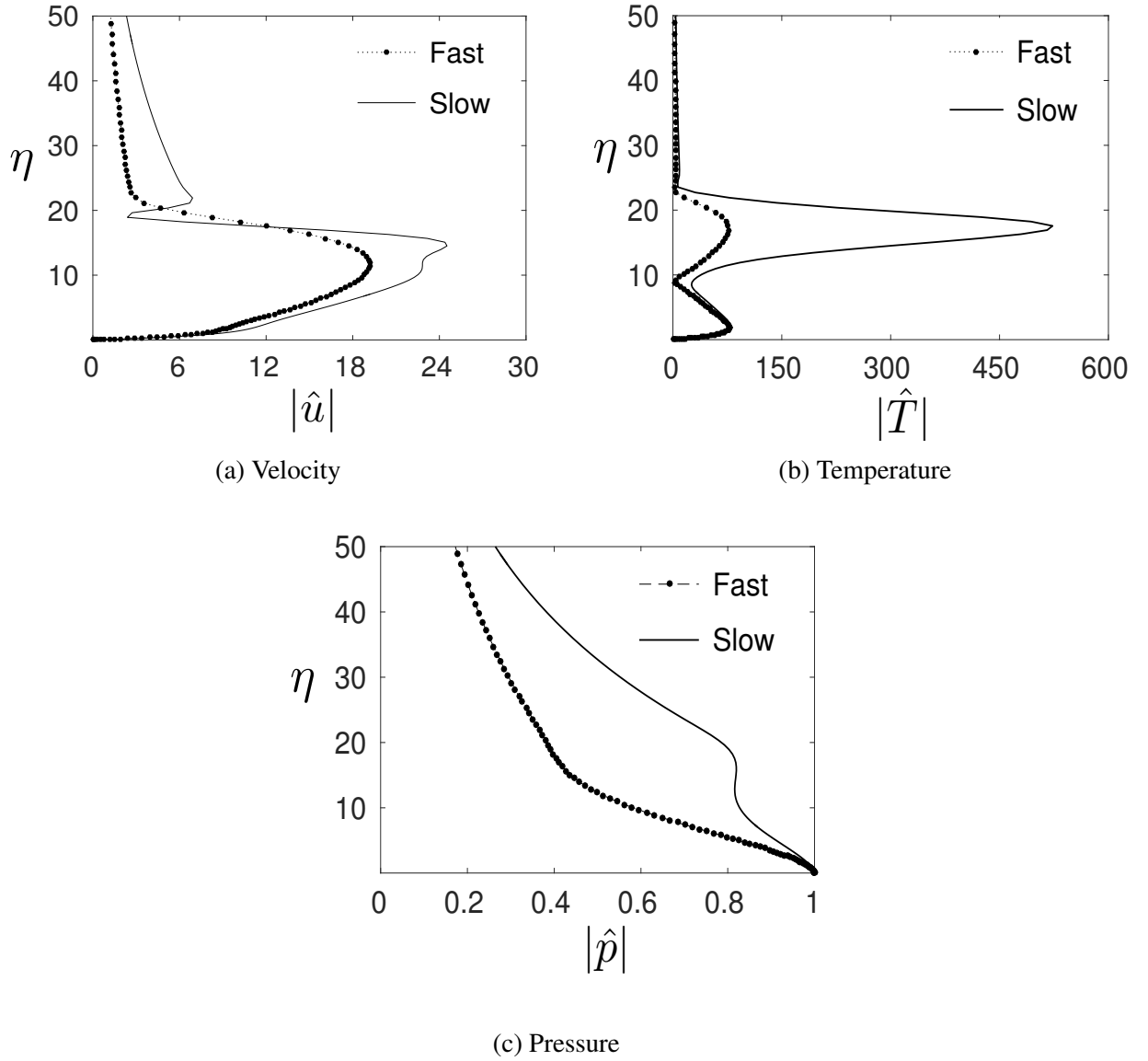


Figure 9.7: Eigenfunctions corresponding to the slow and fast modes (F^-) at $M = 6$, $Re = 3000$, $T_o = 300\text{K}$, $\alpha = 0.05$, $\beta = 0$ and $Pr = 0.4$.

The normalized eigenfunctions for F^- ($c_r = 1 + 0$) and F^+ ($c_r = 1 - 0$) are presented in Fig. 9.8. Within the boundary layer, the eigenfunctions of both the fast modes have a similar structure, however, at the free-stream, they show different characteristics for velocity and temperature eigenfunction. This has been attributed to the interaction of the fast modes with the vorticity and entropy branch of the continuous spectrum in Ref. [69].

As the wavenumber is increased to 0.1009, the fast mode F^+ and the slow mode attain a phase speed close to one another. To study the effect of synchronization, we have plotted the eigenfunctions corresponding to these two discrete modes in Fig. 9.9. Near the wall, we observe an overlap of mode characteristics. Since the wavenumber value does not exactly correspond to the branch point, small differences are observed in the eigenfunctions away from the wall. When the disturbance wavenumber is increased slightly to 0.122, the slow mode attains its

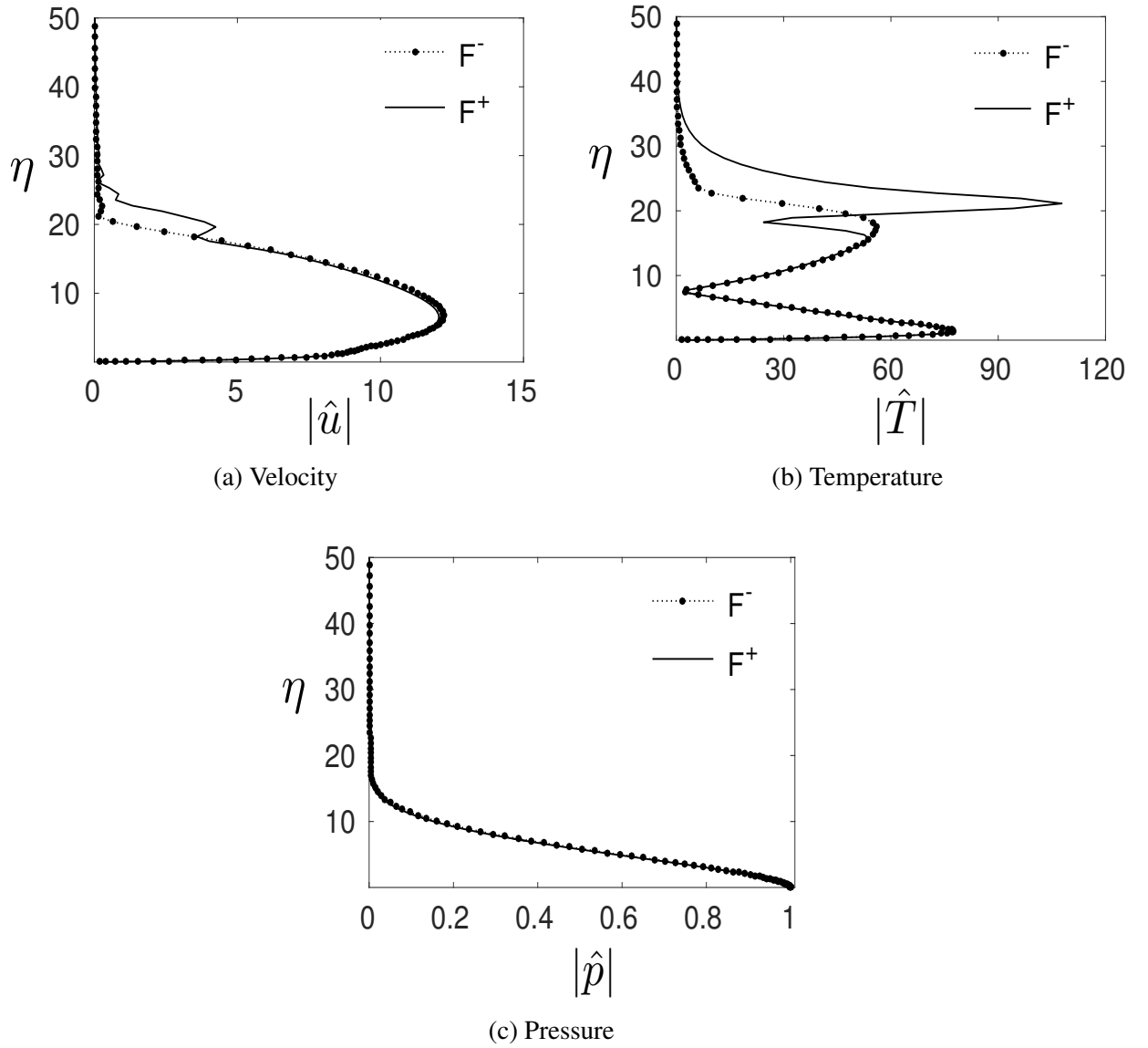


Figure 9.8: Eigenfunctions for the fast modes at $M = 6$, $Re = 3000$, $T_o = 300K$, $\beta = 0$ and $Pr = 0.4$, in the vicinity of the continuous vorticity and entropy branch. The wavenumbers considered for plotting the eigenfunctions for the modes F^- and F^+ are 0.094 and 0.097 respectively.

maximum growth rate due to the synchronization with mode F^+ . Since the phase speed of both the modes at this wavenumber is close, eigenfunctions of both the modes behave in a similar manner, except near the critical layer which is around a wall-normal location of 20.

9.3.6 Effect of Prandtl number on the critical Reynolds number

Fig. 9.11 plots the contours of zero and positive growth rates of the most unstable mode in the $Re - \alpha$ plane for a $M = 6$ flow at three different Prandtl numbers. The reference temperature is 288K. In the case of $Pr = 0.56$ and 0.7, we observe two distinct loops of instability, one appears at low wave numbers and the other occurs at higher wave numbers. The lower loop and

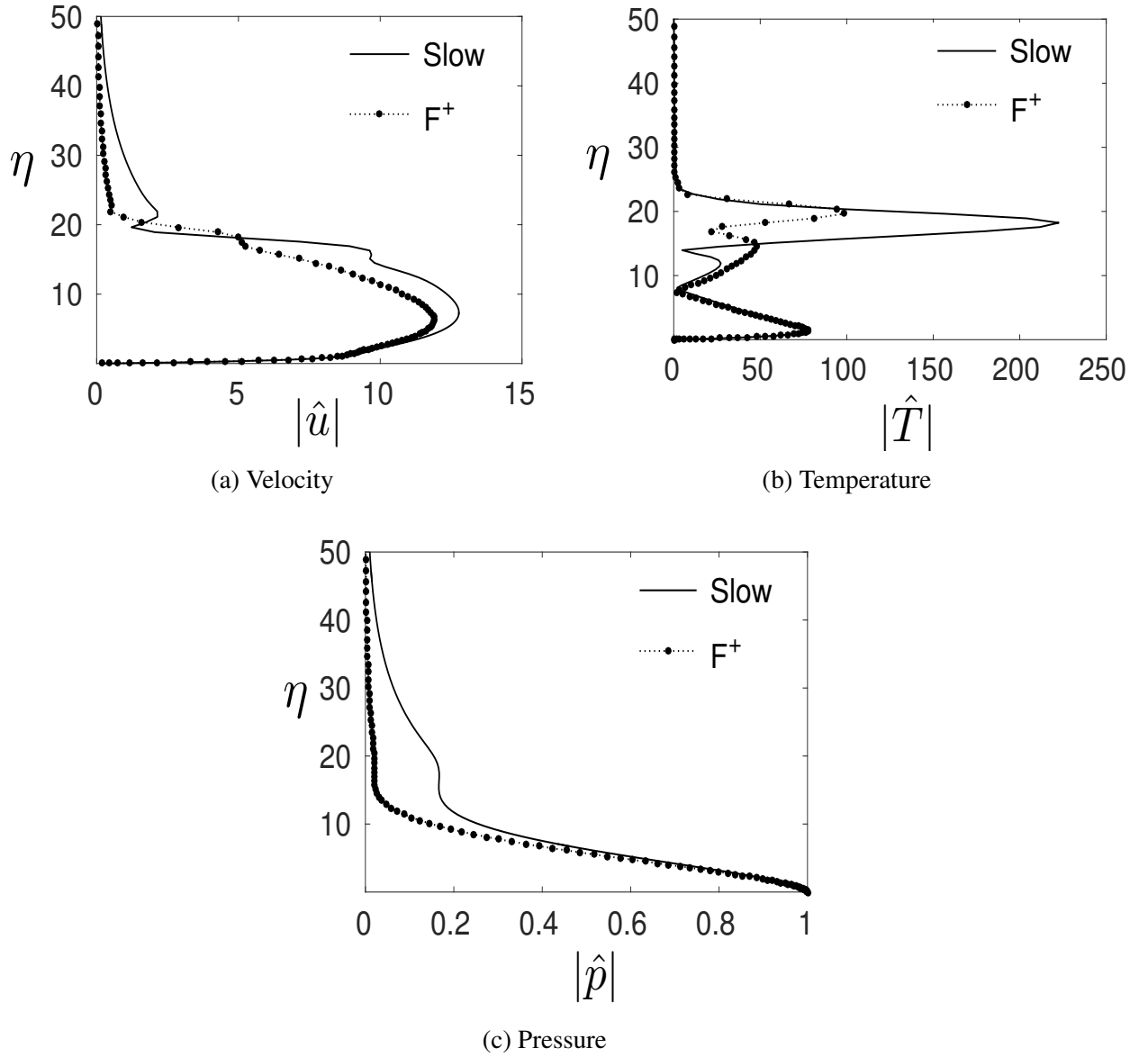


Figure 9.9: Eigenfunctions in the vicinity of the synchronization (branch point) between the slow and fast mode (F^+) at $M = 6$, $Re = 3000$, $T_o = 300\text{K}$, $\beta = 0$, $Pr = 0.4$ and $\alpha = 0.1009$.

the upper loop instability are due to the first and second mode respectively [73]. At a higher Prandtl number of 0.9, the two loops merge to form one large region of instability. The traces of the two modes can still be seen by noting that there are two loops with a growth rate of 0.001 (see Fig. 9.11c).

Overall, the regions of instability corresponding to both the loops increase with an increase in the Prandtl number. However, the first mode is strongly destabilized due to an increase in Prandtl number from 0.56 to 0.7. The first loop shifts to a much lower Reynolds number, resulting in higher growth rates at each Reynolds number with increasing Prandtl number. A further destabilizing effect can be seen for this loop when it merges with the second loop for $Pr = 0.9$. The maximum growth rate of the first mode increases by two orders of magnitude,

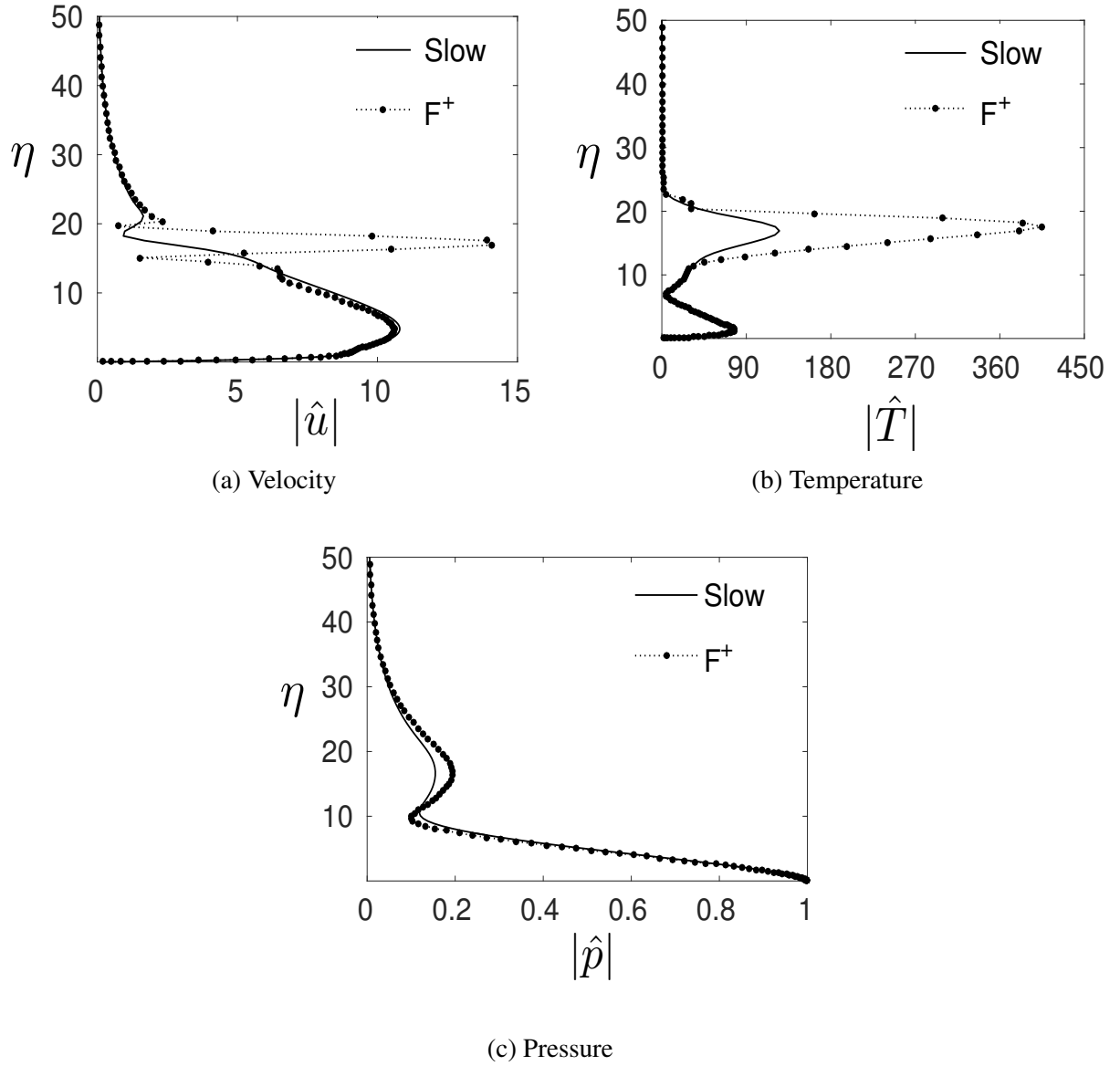


Figure 9.10: Eigenfunctions at $\alpha = 0.122$, corresponding to the maximum peak/trough in the growth rate during synchronization between the slow and fast mode (F^+) at $M = 6$, $Re = 3000$, $T_o = 300K$, $\beta = 0$ and $Pr = 0.4$.

from around 10^{-5} to 10^{-3} , when the Prandtl number changes from 0.56 to 0.9 in the range of Reynolds numbers considered.

The effect of a change in the Prandtl number on the second mode instability is less pronounced in the sense that the characteristics of the second loop are less affected. Though the region of instability increases for the second mode when Pr changes from 0.56 to 0.9, the maximum growth rate only changes from 0.0025 to 0.004.

Given the ambiguity in the terminology of the first and second modes pointed out in Ref. [69], we try to relate the first and the second mode instability to the fast and the slow mode. At $M = 6$ and for all the Prandtl number cases considered in Fig. 9.11, the maximum amplification rate in both the loops corresponds to the slow mode instability. Therefore, an increase in the Prandtl

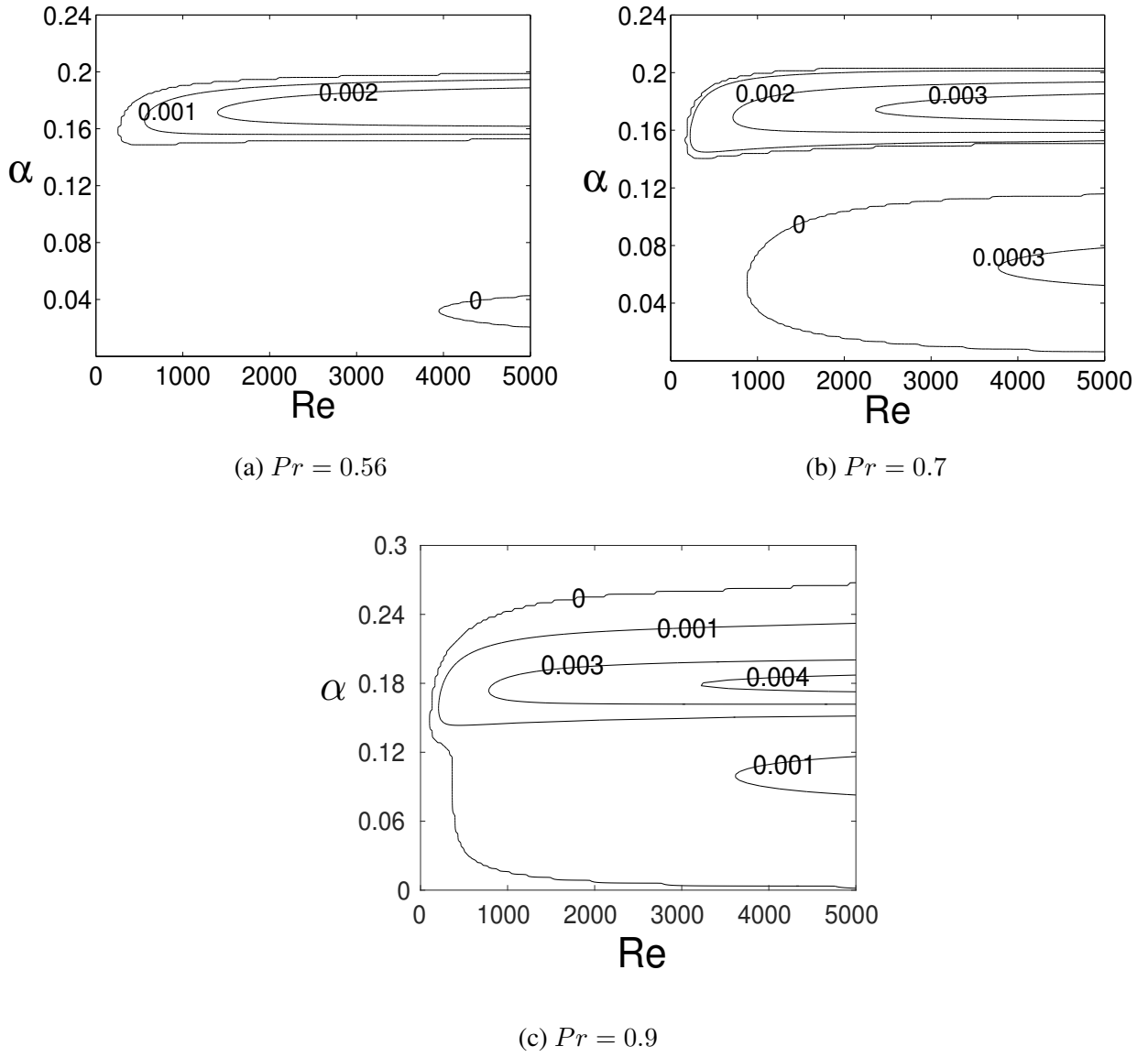


Figure 9.11: Effect of Prandtl number on the least stable mode in the Reynolds number vs. wavenumber plane at $M = 6$.

number destabilizes the slow mode, which is more pronounced at low wavenumbers (in the bottom loop).

Note that in the case of $M = 4$ and all the Prandtl numbers presented in Fig. 9.4, the branching pattern is such that slow and fast modes correspond to the first and second modes respectively. Even in this case, it can be seen that the second mode instability is less affected by a change in the Prandtl number value compared to the first mode instability.

Therefore, we can infer that the stability of the first mode (slow mode) is sensitive to the changes in the Prandtl number, while the second mode (fast or slow mode depending on the branching pattern) is less affected by the Prandtl number variation. Note however that an increase in the Prandtl number destabilizes both the first and second mode.

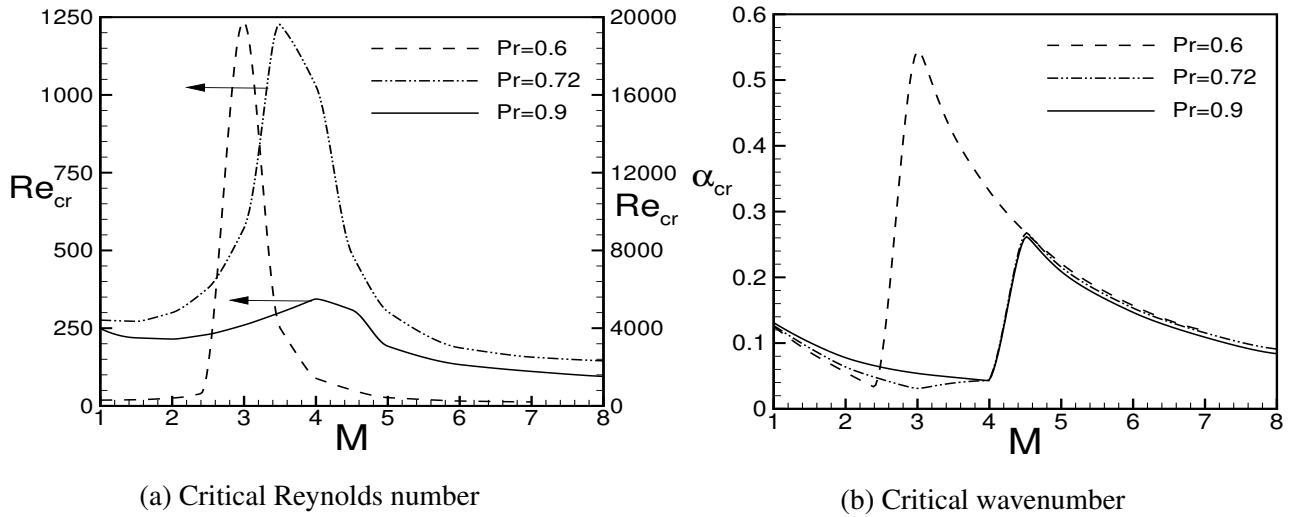


Figure 9.12: Variation of critical Reynolds number and corresponding wave number with Mach number at three different Prandtl numbers 0.6, 0.72 and 0.9.

Fig. 9.12 shows the variation of the critical Reynolds number and wave number with Mach number at three Prandtl numbers 0.6, 0.72 and 0.9. We notice that as the Prandtl number is decreased from 0.9 to 0.72, critical Reynolds number decreases significantly. This effect is more prominent at intermediate Mach numbers (between 2 and 5). For example, at Mach number of 4, the critical Reynolds number reduces by a factor of 3. This decrease in the critical Reynolds number is caused by the slow mode instability (lower loop) as shown in Fig. 9.11. On the other hand, a decrease in the Prandtl number from 0.72 to 0.6 can lead to an increase in the critical Reynolds number by around 35 times at these intermediate Mach numbers. For example, at $M = 3$, the critical Reynolds number which is 572 at $Pr = 0.72$ increases to a value of 19883 at $Pr = 0.6$. This increase in the critical Reynolds number is caused by the fast mode instability after it crosses the entropy and the vorticity branch. As the Mach number increases ($= 4$), the synchronization between the slow and the fast mode leads to the most unstable second mode instability. As the Mach number increases further, the Reynolds number at which these two modes synchronize comes down and hence we observe a decrease in the critical Reynolds number.

From Fig. 9.12b, we observe that between a Mach number of 4 and 5, there is a sharp change in the wave number corresponding to the critical Reynolds number for the case of $Pr = 0.72$ and 0.9. This jump in the critical wave number is due to the change in the most unstable mode, from first to the second mode. The first mode is the dominant instability at low Mach numbers, while the second mode becomes unstable at higher Mach numbers (≥ 4). We notice that the

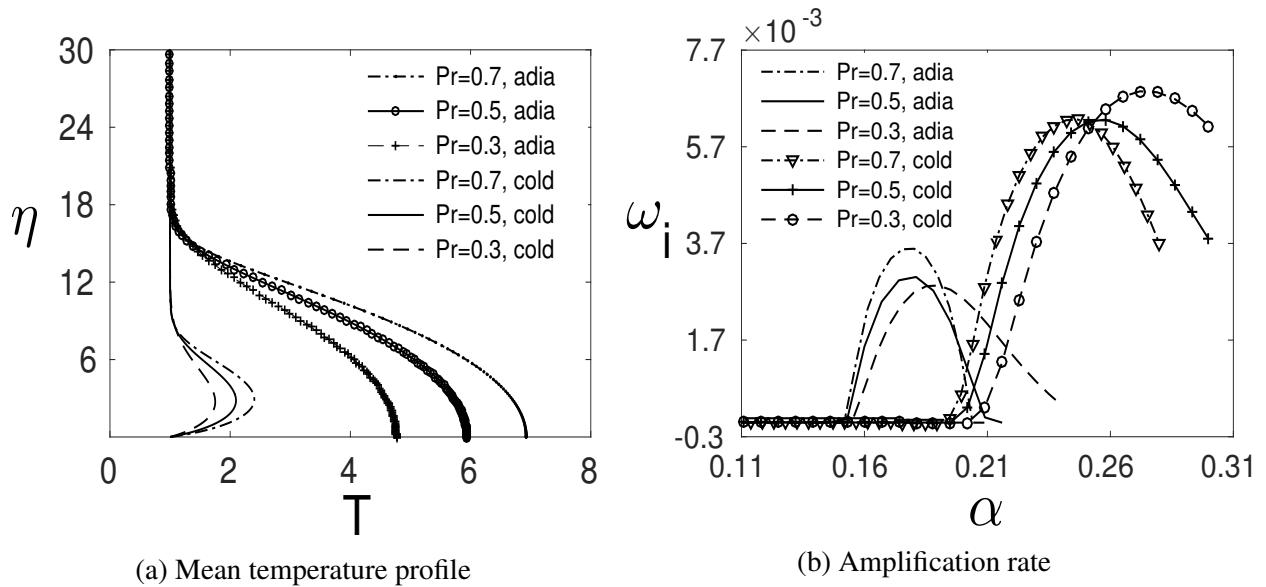


Figure 9.13: (b) Comparison of the growth rate of the second instability mode for a flat plate boundary layer maintained at a cold wall vs. an insulated condition. The mean temperature profiles at different Prandtl number and wall boundary conditions are shown in (a). Mach number and Reynolds number considered are 6 and 5000 respectively. The wall temperature ratio ($\frac{T_w}{T_{ad}}$) for $Pr = 0.7, 0.5$ and 0.3 are 0.14, 0.17 and 0.21 respectively.

first mode is always the slow mode, while the second mode could either be the slow or the fast mode depending on the branching pattern (see section 9.3.4). For the case of $Pr = 0.6$, the fast mode becomes unstable at a higher wavenumber (after it synchronizes with the vorticity/entropy branch) at a Mach number of 3. As the Mach number is increased, the synchronization between the fast and slow mode brings down the critical number to a lower value.

9.3.7 Effect of wall-cooling at different Prandtl numbers

In this section, the effect of wall-cooling on the second mode instability will be examined. In addition, we will study the effect of Prandtl number when the adiabatic boundary condition is replaced with a cold wall value. For this, a Mach 6 flow is considered over a flat plate at a Reynolds number of 5000. Both adiabatic and cold wall boundary conditions are applied at the flat plate surface and the corresponding mean temperature profiles are plotted in Fig. 9.13a. The effect of cooling is examined at three different Prandtl numbers: 0.7, 0.5 and 0.3.

Now, let's consider the case of $Pr = 0.3$. As we can see from Fig. 9.13a, when the boundary condition is changed from adiabatic to the cold wall ($\frac{T_w}{T_{ad}} = 0.21$), the peak temperature in the boundary layer decreases significantly. In addition, cooling the wall reduces the boundary layer thickness considerably. Also, we notice that the peak growth rate of the second mode appears at a much higher wavenumber for the cold wall case. These two observations are valid for all

the Prandtl numbers presented in Fig. 9.13.

The destabilizing effect of cooling can be related to the corresponding reduction in the boundary layer thickness. It is well established that the second mode wavelength is twice the boundary layer thickness. Since cooling decreases the thickness of the mean temperature profile, it leads to an increase in the second mode frequency, in turn increasing its amplitude. Therefore, cooling the surface of the flat plate increases the amplification rate of the second mode and hence makes it more unstable.

Next, we will discuss the effect of the Prandtl number for two different boundary conditions (cold and adiabatic) imposed at the flat plate surface. In the case of an adiabatic wall, it can be observed that as the Prandtl number decreases, the wall temperature also reduces, since the recovery factor scales as the square root of Prandtl number (see Fig. 9.13a). The variation of the growth rate with wavenumber shows that increasing the Prandtl number makes the second mode more unstable for the mean flow profiles with an insulated boundary condition (see Fig. 9.13b). This can be attributed to the decrease in the relative supersonic region with increasing Prandtl number in the case of adiabatic boundary layers, which leads to an increase in the second mode frequency, resulting in a higher amplification rate.

Now, examining the cold wall cases, we notice that the peak temperature reduces with a decrease in the Prandtl number value similar to the flow with adiabatic boundary condition (see Fig. 9.13a). However, the growth rate plot shows that a change in the Prandtl number has a different effect on the second mode instability in the case of cold wall boundary layer compared to an insulated profile. In this case, decreasing the Prandtl number destabilizes the second mode and shifts the peak growth rate to a higher wavenumber (see Fig. 9.13b), which is different from the adiabatic wall case. However, the physical mechanism leading to instability remains same, i.e., decreasing the Prandtl number reduces the relative supersonic region, resulting in an increase in the second mode frequency, leading to an enhanced instability.

9.4 Summary

In this chapter, we investigated the sensitivity of a high-speed boundary layer to small changes in the Prandtl number of the flow. Prandtl number values are systematically varied and a temporal linear stability analysis is performed. We find that increasing the Prandtl number has

a destabilizing effect on the flow at all Mach and Reynolds numbers. The critical Reynolds number has significantly increased with a decrease in the Prandtl number value, especially at intermediate Mach numbers (2 to 5). Higher growth rates and larger regions of instability are observed in the $Re - \alpha$ stability diagrams as the Prandtl number is increased. Two types of branching patterns are observed at the mode-synchronization depending on the Prandtl number, which leads to either the slow or the fast mode becoming the dominant instability. Increasing the Prandtl number destabilizes both the first and the second mode. We observe that the first mode is always the slow mode, while the second mode could either be the slow or the fast mode depending on the branching pattern. Towards the end of the chapter, we also discussed the effect of wall-cooling at different Prandtl numbers and found that cooling the wall can increase the amplification rate of the second mode significantly.

Chapter 10

Summary

In realistic high Mach number flows, transport properties have large independent variations with temperature and pressure, which implies that they can not be related using a constant Prandtl number. In this report, we carry out a series of numerical experiments to isolate the effect of transport property variation on high-speed flow stability, using linear stability theory and transient growth analysis. For this, three different model Couette flows are considered, namely, the reference flow ($\bar{\mu} = \bar{\kappa} = 1$), flow with stratified viscosity and conductivity, and another flow where either viscosity or thermal conductivity is varying, keeping the other constant across the shear layer. All the above flows are maintained at the same reference conditions. To isolate the effect of the gradients in transport properties from their magnitude, we define an equivalent uniform flow, with the same average values of transport properties as that of the corresponding stratified flow.

Temporal linear stability theory is applied for a large range of Mach numbers and Reynolds numbers, for different values of reference Prandtl number. Cases with Prandtl number varying across the shear layer are also considered. The growth rates of the most unstable eigenmode and the critical Reynolds number are reported for the different base flows. The results indicate a significant destabilizing role of increasing the mean conductivity value, while the gradient in conductivity has a mixed stabilizing and destabilizing effect depending on the Mach and Prandtl number of the flow. By comparison, the gradient in viscosity has a strong stabilizing role at all Mach, Reynolds, and Prandtl numbers. The stabilizing effect increases with Mach number, and it can lead to an order of magnitude increase in the critical Reynolds number of the flow. The simultaneous variation in viscosity and conductivity is also found to stabilize all the model flows

considered in this work, where the physical effect of viscosity stratification dominates over that of thermal conductivity variations.

To investigate the effect of transport properties on short time flow dynamics, we compute the maximum transient energy amplification for a wide range of disturbance wavenumbers and mean flow parameters. The Mack energy norm for compressible flows is used. Transient growth analysis reveals similar trends as predicted by the linear stability theory, namely, a significant reduction in energy amplification due to the presence of viscosity gradients in the flow, and a considerable increase in the disturbance energy amplification for the higher thermal conductivity of the fluid. Flows with either viscosity or conductivity stratification, the optimal transient energy gain is reached for $\alpha = 0$, i.e., streamwise independent disturbances. These are initially streamwise vortices that evolve into low and high-speed streaks in streamwise velocity. This corresponds to the well-known lift-up effect and results in alternating streaks of high and low temperature and density fluctuations. The perturbation energy is found to have a large temperature contribution, with an appreciable exchange between the internal and kinetic energy of the perturbations.

Stratification of both viscosity and conductivity decreases the transient energy growth, and the extent of energy amplification decreases with an increase in the Mach number. The optimal disturbance, in this case, has non-zero streamwise wavenumber; the vortical structures are not streamwise independent and are tilted against the mean shear. In this case, the disturbances primarily gain energy using the vortex-tilting mechanism but obtain an additional increment of growth due to the energy transfer from the mean flow, which occurs via Reynolds stress, and the coupling of the wall-normal velocity with temperature and density fluctuations due to compressibility. The energy is extracted from the mean shear by transporting momentum down the mean momentum gradient (through Orr-mechanism) while trying to rise to an upright position as time evolves.

To get an insight into the underlying physical processes leading to the transient energy growth, we present a non-modal energy budget for the total perturbation energy. The variation of constituent energies with time shows that the production, dissipation and thermal diffusion are the main contributors to the total disturbance energy. We find that both the energy transfer from the mean flow and the energy lost by the viscous forces increase significantly with an increase in the conductivity value, leading to a higher transient energy gain in the κ -averaged

Table 10.1: Effect of transport property variation for a compressible Couette flow.

Transport properties	Increase in magnitude	Gradient
Viscosity	Stabilizing	Stabilizing
Thermal conductivity	Strongly destabilizing	Mixed effect depending on Mach and Prandtl number
Viscosity and thermal conductivity	Destabilizing	Stabilizing
Prandtl number	Stabilizing	Stabilizing

flow. Stratification in viscosity has the opposite effect on the constituent energies and results in a lower transient energy amplification. There is a significant reduction in the production term when both viscosity and conductivity stratification is present in the flow, causing a smaller transient energy growth. In addition, the energy amplification occurs over a smaller time scale in μ - κ -stratified Couette flows.

We find that both the effect of variation in viscosity and conductivity can be quantified in terms of a single non-dimensional number, which is the Prandtl number of the flow. The increase in conductivity or decrease in the viscosity value can be interpreted as the reduction in the Prandtl number value. The strong destabilizing role of decreasing the Prandtl number can be attributed to the increase in the thermal diffusivity or decrease in the momentum diffusivity in the flow.

Next, we analyzed the sensitivity of a high-speed boundary layer flow over a flat plate to the changes in the Prandtl number. In contrast to the compressible Couette flow, we find that decreasing the Prandtl number has a significant stabilizing effect in this case. The critical Reynolds number is significantly increased with a decrease in the Prandtl number especially at intermediate Mach numbers (2 – 5). Higher growth rates and larger regions of instability are observed in the $Re - \alpha$ stability diagrams as the Prandtl number is increased. Two types of branching patterns are observed at the mode-synchronization depending on the Prandtl number, which leads to either the slow or the fast mode becoming the dominant instability. Increasing the Prandtl number is found to destabilize both the first and the second mode instability.

In summary, we can infer that the stability of high-speed shear flows is a strong function of the conductivity and viscosity of the fluid, as well as their gradients across the shear layer. Individual stabilizing and destabilizing effects compete with each other for a given Mach and Reynolds number, and the net effect of viscosity and conductivity variations can be captured

by a change in the Prandtl number. The stability effects are also geometry dependent, and not trivially extendable from one flow configuration to another. However, these simple shear flows can help us to study and understand different physical effects, which may not be possible in a real experiment. The insights gained in the process, we believe, will provide useful input towards an understanding of the stability of high-speed flows.

Contribution

1. To the best of our knowledge, the effect of variation in thermal conductivity on flow stability in the case of compressible flows has not been reported before. We examined the effect of a change in the magnitude of conductivity and also its gradient using linear stability theory and transient growth. It is found that increasing the conductivity can have a strong destabilizing effect, and gradient in the conductivity has a negligible role on flow stability.
2. Although the effect of viscosity stratification has been investigated widely, it is often mixed with the effect of thermal conductivity, since the variation of both is related through a constant Prandtl number. In this work, we have isolated the variation of viscosity from conductivity and studied the individual effect of the magnitude of viscosity and its gradient. Both the short-term and asymptotic results show that gradient in viscosity has a strong stabilizing effect.
3. In a realistic flow, where both viscosity and conductivity vary simultaneously, we find that the physical effect of viscosity stratification dominates over that of thermal conductivity variations.
4. We observe that a single non-dimensional number, which is the Prandtl number of the flow can capture the variation of both viscosity and thermal conductivity. In the case of compressible Couette flow, it is noticed that decreasing the Prandtl number can significantly delay the transition process, which in turn can be attributed to the increase in the thermal diffusivity or decrease in the momentum diffusivity in the flow.
5. We also determined a threshold Prandtl number corresponding to a fixed Reynolds num-

ber, which can be used to identify the most unstable mode (mode 1 or 2) in a compressible Couette flow.

6. A change in the Prandtl number is also found to have a dominating role in the modal stability of a flat plate boundary layer. We find that increasing the Prandtl number can strongly destabilize the boundary layer with an adiabatic boundary condition.

Future work

In this work, we have studied the effect of variation of transport properties for a perfect gas flow. However, the temperature in a high enthalpy flow can be significantly higher. This can lead to a vibrationally excited or even a thermo-chemical non-equilibrium flow. It is important to quantify the effect of transport properties in the presence of these physio-chemical effects. The effect of viscosity in the case of a vibrationally excited Couette flow has been recently studied by Grigorev and Ershov [95]. They found that, for the entire range of Mach numbers considered, thermal relaxation exerts a stabilizing effect on the flow and depending on the degree of thermal non-equilibrium, the critical Reynolds number may exceed the corresponding value for a perfect gas by 12%. Similar to the perfect gas results presented in our work, stabilizing effect of viscosity is also observed for a vibrationally excited gas, and it will be useful to examine its effect when the flow is in thermo-chemical non-equilibrium.

It will also be interesting to investigate the effect of variation in the transport properties on the non-linear stages of transition. The different processes leading to a laminar to turbulent transition, for example, the formation of streaks, their breakdown, development of turbulent spots and the laminar-turbulent patterns can be affected significantly by a change in the transport properties. Therefore, a controlled study should be carried out to examine each of these processes by varying different transport properties.

Appendix I: Linear stability equations

The linearized equations for the perturbations are mentioned below.

Continuity equation,

$$\frac{\partial \tilde{\rho}}{\partial t} + \bar{U} \frac{\partial \tilde{\rho}}{\partial x} + \tilde{v} \frac{\partial \tilde{\rho}}{\partial y} + \bar{\rho} \left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} \right) = 0. \quad (\text{AI. 1})$$

x-momentum equation,

$$\begin{aligned} \bar{\rho} \left(\frac{\partial \tilde{u}}{\partial t} + \bar{U} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{d\bar{u}}{dy} \right) = & -\frac{1}{\gamma M^2} \frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re} \frac{\partial}{\partial x} \left[(\bar{\lambda} + 2\bar{\mu}) \frac{\partial \tilde{u}}{\partial x} + \bar{\lambda} \left(\frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} \right) \right] + \\ & \frac{1}{Re} \frac{\partial}{\partial y} \left[\bar{\mu} \frac{d\bar{U}}{dy} + \bar{\mu} \left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x} \right) \right] + \frac{\bar{\mu}}{Re} \frac{\partial}{\partial z} \left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial x} \right). \end{aligned} \quad (\text{AI. 2})$$

y-momentum equation,

$$\begin{aligned} \bar{\rho} \left(\frac{\partial \tilde{v}}{\partial t} + \bar{U} \frac{\partial \tilde{v}}{\partial x} \right) = & -\frac{1}{\gamma M^2} \frac{\partial \tilde{p}}{\partial y} + \frac{1}{Re} \frac{\partial}{\partial y} \left[(\bar{\lambda} + 2\bar{\mu}) \frac{\partial \tilde{v}}{\partial y} + \bar{\lambda} \left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{w}}{\partial z} \right) \right] + \\ & \frac{1}{Re} \frac{\partial}{\partial x} \left[\bar{\mu} \frac{d\bar{U}}{dy} + \bar{\mu} \left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x} \right) \right] + \frac{\bar{\mu}}{Re} \frac{\partial}{\partial z} \left(\frac{\partial \tilde{v}}{\partial z} + \frac{\partial \tilde{w}}{\partial y} \right). \end{aligned} \quad (\text{AI. 3})$$

z-momentum equation,

$$\begin{aligned} \bar{\rho} \left(\frac{\partial \tilde{w}}{\partial t} + \bar{U} \frac{\partial \tilde{w}}{\partial x} \right) = & -\frac{1}{\gamma M^2} \frac{\partial \tilde{p}}{\partial z} + \frac{1}{Re} \left[\bar{\mu} \frac{\partial}{\partial x} \left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial x} \right) + \frac{\partial}{\partial y} \bar{\mu} \left(\frac{\partial \tilde{v}}{\partial z} + \frac{\partial \tilde{w}}{\partial y} \right) \right] + \\ & \frac{1}{Re} \frac{\partial}{\partial z} \left[(\bar{\lambda} + 2\bar{\mu}) \frac{\partial \tilde{w}}{\partial z} + \bar{\lambda} \left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} \right) \right]. \end{aligned} \quad (\text{AI. 4})$$

Energy equation,

$$\begin{aligned} \bar{\rho} \left(\frac{\partial \tilde{T}}{\partial t} + \bar{U} \frac{\partial \tilde{T}}{\partial x} + \tilde{v} \frac{\partial \tilde{T}}{\partial y} \right) = & \frac{\gamma(\gamma-1)M^2}{Re} \left[2\bar{\mu} \frac{d\bar{U}}{dy} \left(\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} \right) + \bar{\mu} \left(\frac{d\bar{U}}{dy} \right)^2 \right] - (\gamma-1) \bar{p} \left(\frac{\partial \tilde{u}}{\partial x} + \right. \\ & \left. \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} \right) + \frac{\gamma}{Re Pr} \left[\bar{\kappa} \left(\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial z^2} \right) + \frac{\partial}{\partial y} \left(\bar{\kappa} \frac{d\bar{T}}{dy} + \bar{\kappa} \frac{\partial \tilde{T}}{\partial y} \right) \right]. \end{aligned} \quad (\text{AI. 5})$$

Ideal gas equation,

$$\tilde{p} = \frac{\tilde{T}}{\bar{T}} + \frac{\tilde{\rho}}{\bar{\rho}}. \quad (\text{AI. 6})$$

The eigenvalue problem is given by,

$$B\psi = \omega I\psi. \quad (\text{AI. 7})$$

Here, $\psi = (\hat{u}, \hat{v}, \hat{w}, \hat{\rho}, \hat{T})$ is the vector of the complex amplitudes of the flow variables and I is identity matrix. B is a $5N \times 5N$ (N is the number of Chebyshev points) complex matrix, and its elements are functions of $\alpha, \beta, \omega, Re, Pr, M$ and the mean-flow variables.

$$\begin{bmatrix} B_{11} & B_{12} & \dots & B_{15} \\ B_{21} & B_{22} & \dots & B_{25} \\ \vdots & \vdots & \ddots & \vdots \\ B_{51} & B_{52} & \dots & B_{55} \end{bmatrix} \begin{Bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{\rho} \\ \hat{T} \end{Bmatrix} = \omega \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{Bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{\rho} \\ \hat{T} \end{Bmatrix} \quad (\text{AI. 8})$$

The rows in the above system correspond to the x-momentum, y-momentum, z-momentum, continuity, and energy equation respectively. The elements of the matrix B are:

$$\begin{aligned} B_{11} &= \alpha \bar{U} - \iota \frac{[\bar{\mu}(\alpha^2 + \beta^2) + \alpha^2(\bar{\mu} + \bar{\lambda}) - \bar{\mu}_y D - \bar{\mu} D^2]}{\bar{\rho} Re}, \\ B_{12} &= -\iota \bar{U}_y - \alpha \frac{[\bar{\mu}_y + (\bar{\lambda} + \bar{\mu}) D]}{\bar{\rho} Re}, \\ B_{13} &= -\iota \alpha \beta \frac{(\bar{\lambda} + \bar{\mu})}{\bar{\rho} Re}, \\ B_{14} &= \frac{\alpha \bar{T}^2}{\gamma M^2}, \\ B_{15} &= \frac{\alpha}{\gamma M^2} + \iota \frac{[\bar{U}_{yy} \bar{\mu}_T + \bar{U}_y \bar{T}_y \bar{\mu}_{TT} + \bar{U}_y \bar{\mu}_T D]}{\bar{\rho} Re}, \\ B_{21} &= -\alpha \frac{[\bar{\lambda}_y + (\bar{\lambda} + \bar{\mu}) D]}{\bar{\rho} Re}, \\ B_{22} &= \alpha \bar{U} + \iota \frac{[(\bar{\mu}_y + \bar{\lambda}_y) D - \bar{\mu}(\alpha^2 + \beta^2) + (\bar{\mu} + \bar{\lambda}) D^2 + \bar{\mu}_y D + \bar{\mu} D^2]}{\bar{\rho} Re}, \\ B_{23} &= -\beta \frac{[\bar{\lambda}_y + (\bar{\lambda} + \bar{\mu}) D]}{\bar{\rho} Re}, \end{aligned}$$

$$\begin{aligned}
B_{24} &= -\iota \frac{(\bar{T}_y + \bar{T}D)}{\bar{\rho}\gamma M^2}, \\
B_{25} &= -\iota \frac{(\bar{\rho}_y + \bar{\rho}D)}{\bar{\rho}\gamma M^2} - \frac{\alpha \bar{U}_y \bar{\mu}_T}{\bar{\rho} Re}, \\
B_{31} &= -\iota \alpha \beta \frac{(\bar{\lambda} + \bar{\mu})}{\bar{\rho} Re}, \\
B_{32} &= -\beta \frac{[\bar{\mu}_y + (\bar{\lambda} + \bar{\mu}) D]}{\bar{\rho} Re}, \\
B_{33} &= \alpha \bar{U} - \iota \frac{[\bar{\mu} (\alpha^2 + \beta^2) + \beta^2 (\bar{\mu} + \bar{\lambda}) - \bar{\mu}_y D - \bar{\mu} D^2]}{\bar{\rho} Re}, \\
B_{34} &= \frac{\beta \bar{T}^2}{\gamma M^2}, \\
B_{35} &= \frac{\beta}{\gamma M^2}, \\
B_{41} &= \alpha \bar{\rho}, \\
B_{42} &= -\iota (\bar{\rho}_y + \bar{\rho}D), \\
B_{43} &= \beta \bar{\rho}, \\
B_{44} &= \alpha \bar{U}, \\
B_{45} &= 0, \\
B_{51} &= \alpha (\gamma - 1) \bar{T} + \iota \frac{2\gamma (\gamma - 1) M^2 \bar{\mu} \bar{U}_y D}{\bar{\rho} Re}, \\
B_{52} &= -\iota \bar{T}_y - \iota (\gamma - 1) \bar{T} D - \frac{2\alpha \gamma (\gamma - 1) M^2 \bar{\mu} \bar{U}_y}{\bar{\rho} Re}, \\
B_{53} &= \beta (\gamma - 1) \bar{T}, \\
B_{54} &= 0, \\
B_{55} &= \alpha \bar{U} + \iota \gamma \frac{[\bar{T}_{yy} \bar{k}_T + \bar{T}_y^2 \bar{k}_{TT} + 2\bar{T}_y \bar{k}_T D - (\alpha^2 + \beta^2) \bar{k} + \bar{k} D^2 + (\gamma - 1) M^2 Pr \bar{U}_y^2 \bar{\mu}_T]}{\bar{\rho} Re Pr}.
\end{aligned}$$

Here, D is the differential operator, and subscripts y and T denote the derivatives with respect to y and $\bar{T}$ respectively.

Appendix II: Decomposition of total perturbation energy

The decomposition of total perturbation energy gives rise to production, viscous dissipation, thermal diffusion, shear work, and pressure energy. The mathematical expression corresponding to the individual terms is mentioned below.

Production:

$$\frac{\partial P}{\partial t} = - \int_0^1 \left[\bar{\rho} \bar{U}_y u_s^+ v_s + \frac{\bar{T}}{\gamma M^2 \bar{\rho}} \bar{\rho}_y \rho_s^+ v_s + \frac{\bar{\rho}}{\gamma(\gamma-1) M^2 \bar{T}} \bar{T}_y T_s^+ v_s \right] dy + c.c. \quad (\text{AII. 1})$$

Viscous dissipation:

$$\begin{aligned} \frac{\partial V}{\partial t} = & -\frac{1}{Re} \int_0^1 [\alpha^2 (\bar{\mu} + \bar{\lambda}) u_s^+ u_s + \bar{\mu} (\alpha^2 + \beta^2) u_s^+ u_s - u_s^+ (\bar{\mu}_y D + \bar{\mu} D^2) u_s - i\alpha \bar{\mu}_y u_s^+ v_s \\ & - i\alpha (\bar{\mu} + \bar{\lambda}) u_s^+ D v_s + \alpha \beta (\bar{\mu} + \bar{\lambda}) u_s^+ w_s - (\bar{U}_{yy} \bar{\mu}_T + \bar{U}_y \bar{T}_y \bar{\mu}_{TT}) u_s^+ T_s - \bar{U}_y \bar{\mu}_T u_s^+ D T_s \\ & - i\alpha \bar{\lambda}_y v_s^+ u_s - i\alpha (\bar{\mu} + \bar{\lambda}) v_s^+ D u_s + \bar{\mu} (\alpha^2 + \beta^2) v_s^+ v_s - (\bar{\mu}_y + \bar{\lambda}_y) v_s^+ D v_s - (\bar{\mu} + \bar{\lambda}) v_s^+ D^2 v_s \\ & - \bar{\mu}_y v_s^+ D v_s - \bar{\mu} v_s^+ D^2 v_s - i\beta (\bar{\mu} + \bar{\lambda}) v_s^+ D w_s - i\alpha \bar{U}_y \bar{\mu}_T v_s^+ T_s - i\beta \bar{\lambda}_y v_s^+ w_s - i\beta \bar{\mu}_y w_s^+ v_s \\ & + \alpha \beta (\bar{\mu} + \bar{\lambda}) w_s^+ u_s - i\beta (\bar{\mu} + \bar{\lambda}) w_s^+ D v_s + \bar{\mu} (\alpha^2 + \beta^2) w_s^+ w_s + \beta^2 (\bar{\mu} + \bar{\lambda}) w_s^+ w_s \\ & - \bar{\mu} w_s^+ D^2 w_s - \bar{\mu}_y w_s^+ D w_s] dy + c.c \end{aligned} \quad (\text{AII. 2})$$

Thermal diffusion:

$$\begin{aligned} \frac{\partial T}{\partial t} = & \frac{\bar{\rho}}{\gamma(\gamma-1) Re Pr M^2} \int_0^1 [\bar{\kappa}_T \bar{T}_{yy} T_s^+ T_s + \bar{T}_y^2 \bar{\kappa}_{TT} T_s^+ T_s + 2 \bar{T}_y \bar{\kappa}_T T_s^+ D T_s \\ & - \bar{\kappa} (\alpha^2 + \beta^2) T_s^+ T_s + \bar{\kappa} T_s^+ D^2 T_s] dy + c.c \end{aligned} \quad (\text{AII. 3})$$

Shear work:

$$\frac{\partial S}{\partial t} = \frac{\bar{\rho}}{\gamma Re} \int_0^1 [2 \bar{\mu} \bar{U}_y T_s^+ Du_s + 2 i \alpha \bar{\mu} \bar{U}_y T_s^+ v_s + \bar{\mu}_T \bar{U}_y^2 T_s^+ T_s] dy + c.c \quad (\text{AII. 4})$$

Pressure energy:

$$\frac{\partial W}{\partial t} = \frac{1}{\gamma M^2} \int_0^1 [p_s^+ Dv_s + v_s^+ Dp_s] dy + c.c \quad (\text{AII. 5})$$

where, $p_s = \bar{\rho} T_s + \bar{T} \rho_s$

Appendix III

Validation study for Couette flow

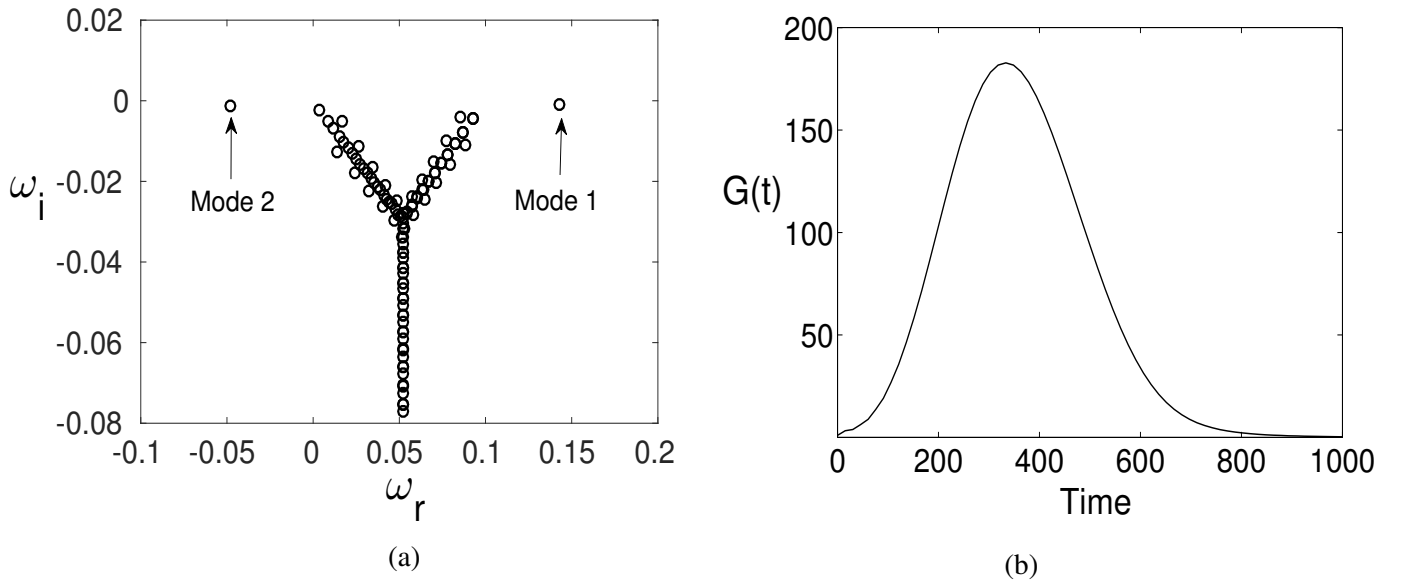


Figure AIII.1: (a) The eigenspectrum magnified near $\omega_r = 0$, for an uniform viscosity and conductivity compressible Couette flow at $M = 2$, $Pr = 0.72$, $Re = 2 \times 10^5$ and $\alpha = \beta = 0.1$. (b) Variation of the energy amplification factor, $G(t)$, with time for a viscosity and conductivity stratified Couette flow, with the same parameters as in (a).

In this section, we carry out a validation study for the compressible Couette flow results. The results are compared with the previous linear stability calculations of Hu and Zhong [20] and Malik, Alam and Dey [67]. Fig. AIII.1a plots the eigenspectrum for a compressible Couette flow with uniform shear at $M = 2$, $Pr = 0.72$, $Re = 2 \times 10^5$ and $\alpha = \beta = 0.1$. The case corresponds to Fig. 2b of Ref. [67]. We notice a good match between our plot (Fig. AIII.1a) and Fig. 2b in Ref. [67]. For the range of disturbance frequencies considered in Fig. AIII.1a, first two acoustic modes are clearly visible in both the figures and their respective growth rates match well. The Y shape of the viscous branch also compares well between the two figures.

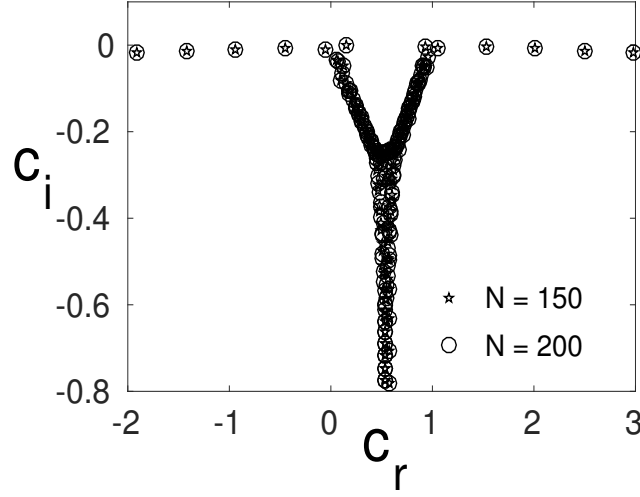


Figure AIII.2: Comparison of the eigenspectrum computed using 150 and 200 chebyshev collocation points for a κ -stratified flow at $M = 10$, $Pr = 0.2$, $Re = 10^5$ and $\alpha = 1$ and $\beta = 0$.

In addition, we have also computed a case from Ref. [20] with parameters $M = 2$, $Pr = 0.72$, $Re = 2 \times 10^5$ and $\alpha = 0.1$. The results computed using 100 Chebyshev points match up to four decimal places with the data mentioned in Ref. [20].

The validation is also performed for transient energy growth calculation, where we compare the variation of energy amplification factor with time for a stratified viscosity and conductivity Couette flow at $M = 2$, $Pr = 0.72$, $Re = 2 \times 10^5$, $\alpha = \beta = 0.1$ (see Fig. AIII.1b). A good match for the maximum energy amplification (~ 180) and the optimal time (~ 330) are found while comparing with Fig. 4a of Ref. [67].

Grid convergence

Fig. AIII.2 plots the real and imaginary parts of the temporal eigenvalues computed for a $M = 10$, $Re = 10^5$ and $Pr = 0.2$, κ -stratified Couette flow at a disturbance wavenumber of $\alpha = 1$ and $\beta = 0$. To test the grid-convergence, we have computed the eigenspectrum using 150 and 200 Chebyshev points. It can be seen that most of the modes in the eigenspectrum are grid-converged especially the acoustic modes. Some of the viscous modes in the tail of the Y shaped region may not be grid-converged, but they do not play an important role in the long-term stability behavior since they are highly damped modes. Therefore, special care has been taken to resolve the acoustic modes accurately, and in this case, 150 Chebyshev points are sufficient to attain the grid-convergence for these modes.

Appendix IV

Validation study for Flat plate boundary layer

This section is dedicated to discuss and validate the stability of high-speed flat plate results in the perspective of experimental and direct numerical simulation (DNS) results. A comparison with the relevant linear stability results is also presented.

Direct numerical simulation along with the parabolized stability equations (PSE) and linear stability analysis plays a vital role in the examination of transition mechanisms. Numerical experiments can lead the way to experimental investigators by systematically identifying the important parameters that can have a dominating effect on the transition process. However, numerical computations can predict very different transition behavior unless the incoming disturbance field and the boundary conditions match closely with the experiment. Compared to the incompressible flows, the transition process at high Mach numbers is not well understood because of the appearance of higher Mack modes and the added complexities of internal mode excitation and chemical reaction. Therefore, computation and experimental works carried out to analyze the transition problem at hypersonic condition is extremely challenging. At hypersonic velocities, it is important to use quiet shock and wind tunnels, and since the high-speed vehicles fly at a low noise environment, the boundary layers on the nozzle walls should be laminar in the experiments. However, such facilities are not easily available, and currently, only three such wind tunnels are available worldwide which can measure a cold Mach 6 flow at a moderate Reynolds number ([97], [98]). It is also not possible for a single wind tunnel to accurately measure all the necessary parameters such as the disturbance level, surface roughness, the Mach number and the enthalpy of the flow [97]. Therefore, we have to combine the knowledge of available theoretical work, numerical simulations, and experimental outcomes to understand the physics of hypersonic transition. Though the flight tests are very expensive and have their

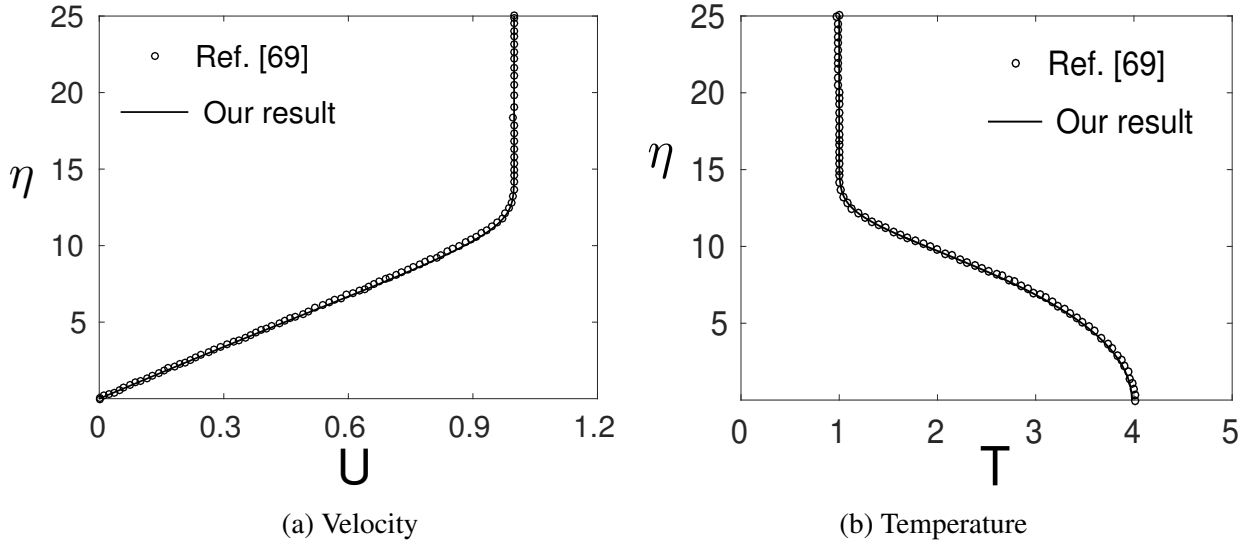


Figure AIV. 1: Variation of mean-flow velocity and temperature profiles for a boundary layer flow at a Mach number of 4.2, $T_o = 300\text{K}$ and $Pr = 0.72$. Our results compare well with the data presented in Fig. 20 of Ref. [69].

own limitations, their results should be used to evaluate the transition prediction models.

The boundary layer cases presented in this report have been validated against a few well-recognized experimental works available in the literature (see section 9.3.2). The second mode, which has been identified as the most unstable instability mode in a hypersonic boundary layer, is known to play a significant role in the laminar to turbulent flow transition. Many experimental works have reported the presence and dominance of this mode ([47], [48]). In our stability analysis, we observe the second mode when the fast and the slow modes synchronize with one another (see section 9.3.4). To visualize this mode, we plot the pressure perturbation contours, which shows the trapped sound waves inside the boundary layer, indicating the presence of Mack's second mode (see Fig. 2.3a). To investigate the role of first and the second mode in the non-linear breakdown process leading to turbulence, Casper et al. [54] and others have carried out many experimental works. While comparing our results with the Schlieren images of Casper et al. [54], we observe a similar structure of the pressure eigenfunction corresponding to the second Mack mode (see Fig. 2.4).

We also compared the boundary layer mean flow profiles and disturbance eigenfunctions with the linear stability calculations of Fedorov and Tumin [69], and found an excellent match. Fig. AIV. 1 plots the mean temperature and velocity profiles for a flow over a flat plate at Mach 4.2. The solid line represents our result, which matches perfectly with the data points taken from Ref. [69], shown by the open circles. The linear stability results computed using the

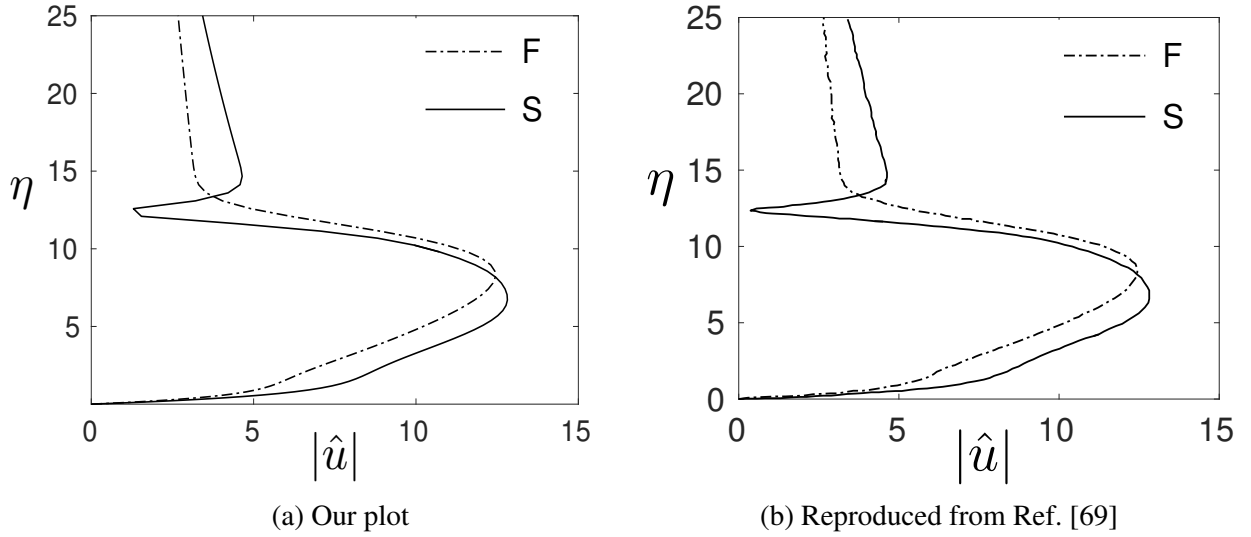


Figure AIV. 2: Velocity eigenfunctions of the fast and slow modes at $M = 4.2$, $Re = 2000$, $Pr = 0.72$, $T_o = 300K$ and $\alpha = 0.05$, corresponding to a flow over a flat plate. The eigenfunctions obtained in our linear stability calculation (a) match well with the figure (b) taken from Ref. [69].

same mean flow profiles are shown in Fig. AIV. 2. The figure plots the velocity eigenfunctions corresponding to the fast and the slow mode at a Reynolds and disturbance wavenumber of 2000 and 0.05 respectively. The eigenfunctions obtained from our linear stability analysis (Fig. AIV. 2a) compare well with the result from Fedorov and Tumin [69] (see Fig. AIV. 2b). In addition, phase velocity plots showing the resonant interactions between the fast and slow mode leading to the generation of Mack modes, are comparable to the numerical computations of Ma and Zhong [99] and Fedorov and Tumin [69].

Stability and transition of a supersonic boundary layer are significantly affected by the heat transfer from the surface [58]. Since a great variety of aerodynamic applications require cooling, the effect of surface cooling has been studied widely in experiments and also in numerical computations ([45], [70], [100]). Bitter and Shepherd [65] investigate the effect of wall-cooling on the amplification rate of the vorticity and the second mode at different Mach number (see Fig. AIV. 3a). Similar to the previous works of Mack and others, they notice that surface-cooling significantly destabilizes the Mack modes and stabilizes the first mode. The experimental work carried out by Masad and Abid [59] on a flat plate boundary layer at supersonic Mach numbers also reveals a similar effect of wall-cooling (see Fig. AIV. 3b). In fact, many other works have reported that till a Mach number of 9, surface-cooling can be used to stabilize the first mode completely ([42], [63], [101]). However, with a reduction in the surface temperature, in addition to an increase in the amplification rate, the location of the peak growth rate of the

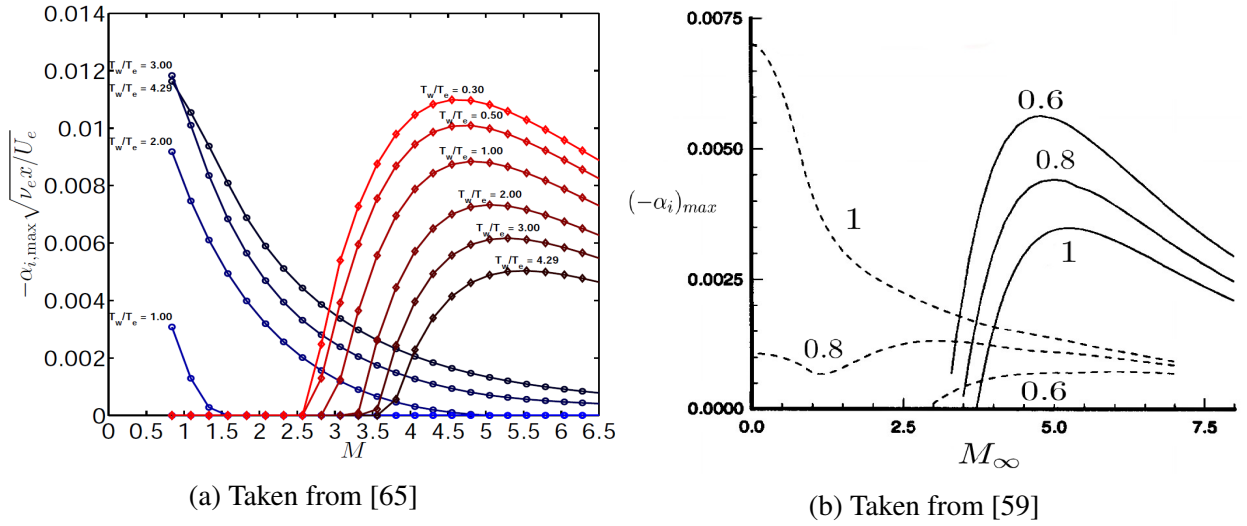


Figure AIV. 3: (a) The effect of wall temperature on spatial growth rates of first and second instability modes at $Re = 1500$. Blue and red curves correspond to the first and second modes respectively. (b) Variation of the maximum growth rate of the instability waves with Mach number for a flat plate boundary layer at $Re = 1500$. The dashed and the solid lines are used for the first and the second mode respectively.

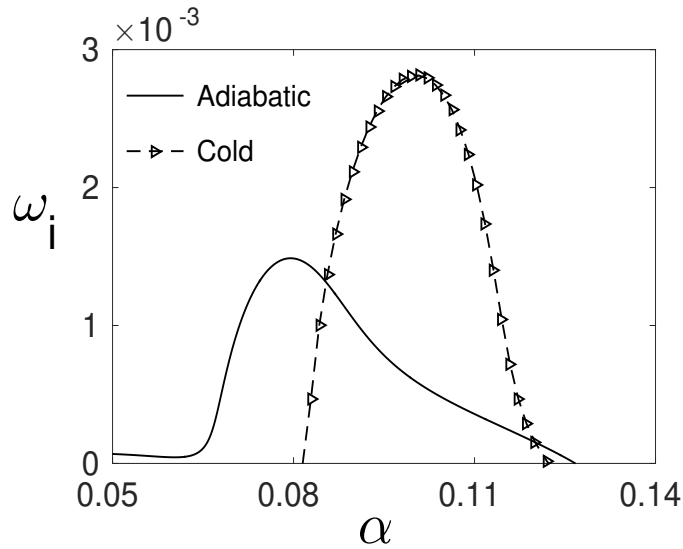


Figure AIV. 4: Comparison of the amplification rate of the second mode at Mach 10, $Re = 1500$ and $Pr = 0.72$. Surface of the flat plate is cooled from adiabatic condition to a temperature ratio of $\frac{T_w}{T_{ad}} = 0.2$.

Mack modes shift to a higher frequency. The results of direct numerical simulation and linear stability analysis show that the growth rate of disturbances is significantly affected by a change in the wall temperature of a blunt cone [100]. Since the effect of cooling is well-known besides its vast practical importance, this is an excellent case to validate with.

To validate our stability results with the previous works related to cooling, we have considered a Mach 10 flow over a flat plate, at a Prandtl and a Reynolds number of 0.72 and

1500 respectively. At the wall, adiabatic as well as cold wall boundary conditions are imposed. Fig. AIV. 4 plots the amplification rate of the second mode as a function of the disturbance wavenumber for both the boundary conditions. We notice that cooling the flat plate surface has increased the peak growth of the second mode significantly. In addition, cooling has also shifted the location of the peak amplification rate to a higher wavenumber value. These effects of wall-cooling are the same as that observed in the previous works [59], [65] (see Fig. AIV. 3).

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List of publications based on present work

Journal publications

1. B. Saikia, A. Ramachandran, K. Sinha and R. Govindarajan, “Effects of viscosity and conductivity stratification on the linear stability and transient growth within compressible Couette flow,” *Physics of Fluids*, Vol 29.2, Jan 2017.
2. A. Ramachandran, B. Saikia, K. Sinha and R. Govindarajan, “Effect of Prandtl number on the linear stability of compressible Couette flow,” *International Journal of Heat and Fluid Flow*, Vol 61, 553-561, 2016.

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1. A. Ramachandran, B. Saikia, K. Sinha and R. Govindarajan, “Linear stability of high-speed boundary layer flows at varying Prandtl numbers,” In 45th AIAA Thermophysics Conference, 2320, 2015.

Published abstracts

1. B. Saikia, A. Ramachandran, K. Sinha and R. Govindarajan, “Effect of Prandtl number on the linear stability of compressible Couette flow,” 68th Annual Meeting of the American Physical Society, Division of Fluid Dynamics, Boston, MA, 22-24 November, 2015.
2. B. Saikia, A. Ramachandran, K. Sinha and R. Govindarajan, “Role of thermal conductivity stratification on the linear stability of a compressible plane Couette flow,” IUTAM Symposium on Advances in Computing of Transitional and Turbulent Flows, Goa, India, December, 2014.